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EDITED BY
T. A. A. BROADBENT, M.A.

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## THE MATHEMATICAL ASSOCIATION

THE Annual Meeting of the Mathematical Association was held at the London Day Training College on 4th-5th January, 1932.

On Monday, 4th January, the proceedings opened at 2.45 p.m., with the transaction of business; Dr. W. F. Sheppard was in the chair. The Report of the Council for the year 1931 was presented and adopted, and the Treasurer's statement of the financial position accepted.

The Chairman announced that the Council had the pleasure of nominating Professor G. N. Watson, Sc.D., F.R.S., for the office of President for the year 1932. He was elected by acclamation. The Officers and Council were re-elected for 1932, with the addition of Sir Arthur Eddington's name to the list of Vice-Presidents, and the substitution of Miss B. Swirles and Mr. C. G. Nobbs for Miss L. M. Swain and Mr. W. C. Fletcher.

Motions to alter certain rules and to form a Branches Sub-Committee were referred back to the Council for consideration.

Sir Arthur Eddington then took the chair, and delivered his Presidential address on *The Decline of Determinism*.* This was followed, after an interval, by Professor Brodetsky's lecture on *Modern Dynamics in Astronomy, Relativity and Quantum Theory*.*

On Tuesday, 5th January, at 10 a.m., a discussion on *The Report of the Association of University Teachers on Entrance Tests and Initial Degrees** was opened by Mr. J. T. Combridge, Miss C. Stimson and Mr. A. C. Heath. This was followed by an address from Dr. Verblunsky on *The Foundations of Mathematics*, after which, the paper by Professor Lodge on *Interpolation by Central Differences* was, owing to his absence, read by Dr. Wishart.

The afternoon meeting opened at 2.30 p.m. with Mr. G. Frecheville's paper on *The Application of Mathematical Methods to Research work in Agricultural Economics*.* The concluding item was a discussion opened by Mr. Trevor Dennis, on *Calculus and Coordinate Geometry at the School Certificate Stage*.*

The usual votes of thanks brought the proceedings to a close. A Publishers' Exhibition was open during the two days.

* To be published later.

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# The Mathematical Association.

## Report of the Council for the Year 1931.

During the year 1931, 86 new members of the Association have been admitted. The number of members now on the roll is 1260, of whom 4 are Honorary members, 75 are Life members by composition, 11 are Life members under the old rule, and 1170 are ordinary members.

The Council regrets to have to record the deaths of Mr. A. A. Bourne, formerly of Cheltenham College; Mr. J. B. Colgrove of Loughborough; the Rev. H. W. Evans, formerly of Bedford School; Mr. W. H. Fletcher of Newington College, Sydney; Mr. J. E. Pearson, formerly of Clifton College; Mr. J. T. Short, of the Secondary School, Pentre, Rhondda; and the Rev. Canon James M. Wilson, D.D.

Canon Wilson and Mr. Bourne were original members of the Association when it was called "The Association for the Improvement of Geometrical Teaching". They helped to form it in 1871 and continued to be members for 60 years. Mr. Colgrove had been a member for 58 years, Mr. Pearson for 54 years and Mr. Evans for 46 years. Canon Wilson served as President of the Association in its Jubilee year, 1921, and in his Presidential Address told the story of the Association's early days.

**The Teaching Committees.**

The General Teaching Committee has met during the year to consider miscellaneous small items of business, and also to discuss the Arithmetic Report prepared by the Boys' Schools Committee. This report is ready for publication. The Boys' Schools Committee is now giving attention to a new Algebra Report.

**Branches.**

The Bristol Branch (members, 16; associates, 27) has held four meetings during the year; a paper on "Loci" by Mr. C. V. Durell roused special interest.

The Cardiff Branch is able to report a most encouraging increase in membership; it has now 22 members and 17 associates. It has held four meetings during the year.

The Liverpool Branch (members, 15; associates, 38) has held five meetings. It is losing its President, Professor Titchmarsh, owing to his appointment to a Chair at Oxford.

The London Branch has during the year increased its membership by 20 members and 18 associates; it now numbers 180 members and 110 associates. Six meetings have been held, with a very encouraging increase in attendance. Early in the year a memorandum on the London University General School syllabus in Mathematics was sent to the University authorities, and in reply the Branch has been informed that in and after 1933 some questions in Trigonometry will be included in that examination.

The Manchester Branch (members, 29; associates, 88) has held

five meetings, the chief topic of interest having been the syllabus of "Additional Mathematics" in the Northern Universities Joint Matriculation Examination. The Branch has drawn up a syllabus and hopes to enlist the support of other Branches in urging its acceptance.

The Midland Branch has nearly doubled its membership during the past year; it now consists of 33 members and 36 associates. The Branch has held four meetings.

The North-Eastern Branch consists of 64 members and 11 associates. It has held four meetings.

The Yorkshire Branch (members, 46; associates, 116) has held three meetings. The Branch has had recently under discussion the regulations of the Northern Joint Matriculation Board. The Annual Dinner was held in the spring term, with Professor E. T. Whittaker as guest.

Detailed reports of the activities of the Branches have either already appeared, or will subsequently appear, in the *Gazette*.

The Council has learned with interest that several members of the Association have expressed a wish that the programme of the Annual Meeting should include a discussion on the character of the mathematics syllabus suitable for Senior (elementary) Schools. While feeling that the Association is as yet hardly in a position to discuss this matter effectively, the Council hopes that the Branches will, as and when opportunity offers, encourage any of their members whose experience entitles them to views on this subject to make them known as a basis for discussion.

**New Branch.**

The Council is pleased to report the formation of a Branch for Southampton and District. Its rules have been laid before the Council and have been approved.

**Library.**

The future of the Library has been under discussion during the year. The present arrangement has been most satisfactory from the point of view of the Association as a whole, and the Council wishes to express its appreciation of the large debt due to Professor Neville and to offer him very cordial thanks for the work he has done. The burden it involves is, however, one which he ought not to be asked to sustain any longer. Various plans have accordingly been discussed and inquiries are being pushed in different directions, but it has not yet been possible for the Council to come to any decision.

The thanks of the Association are due to Mr. F. Beames for the assistance which he has continued to give to the Librarian.

**The Mathematical Gazette.**

With the number to be issued in February, 1932, the *Gazette* will appear in an improved form. The size of the page will not be altered, but the type used will be larger. This change will be accompanied by another—instead of six numbers each year as

hitherto, only five will be issued, consisting of about 80 pages each.

Mr. Broadbent has now completed his first year of office as Editor of the *Gazette*, and the Council desires to express its appreciation of the work he has done and to acknowledge the indebtedness of the Association to him.

**The Problem Bureau.**

The Problem Bureau has continued to carry on its work successfully, but the number of problems sent in has been rather fewer than in previous years. Mr. H. G. Forder has kindly consented to act as an assistant to the Bureau. The Council feels that many members would like to express their grateful thanks to Mr. Gosset Tanner, and those associated with him, for their timely help.

Professor Sir Arthur Eddington retires after this meeting from the office of President which he has held for two years, and the Council desires to record its high appreciation of the services which he has rendered to the Association during his period of office. The Council nominates Professor G. N. Watson, Sc.D., F.R.S., Professor of Mathematics in the University of Birmingham, for election as his successor in the office of President for the year 1932.

The Council nominates Professor Sir Arthur Eddington for election as a Vice-President of the Association.

The Council desires also to express the thanks of the Association to Miss L. M. Swain and Mr. W. C. Fletcher, who now retire from the Council, for the services which they have rendered to the Association during their tenure of office as members of the Council since 1927 and 1928, respectively.

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## GLEANINGS FAR AND NEAR.

**831.** . . . A friend at Court . . . told me that His Majesty's mathematicians, having taken the height of my body by the help of a quadrant, and finding it to exceed theirs in the proportion of twelve to one, they concluded from the similarity of their bodies that mine must contain at least 1724 of theirs.—J. Swift, *Voyage to Lilliput*, chap. iii, last paragraph.

**832.** From Dr. Johnson's *A Journey to the Western Islands*, 1775. After being hospitably entertained at a small village, Dr. Johnson presented the daughter of his host with a book. The following footnote occurs at the bottom of the page on which the presentation is recorded: "The book was Cocker's *Arithmetic*! When twitted with this, Johnson remarked 'They never adverted that I had no *choice* in the matter. I have said that I presented her with a book that I *happened* to have about me'. Boswell ventured to say, 'But, sir, is it not somewhat singular that you should *happen* to have Cocker's *Arithmetic* about you on the journey? What made you buy such a book at Inverness?' 'Why, sir,' replied Johnson, 'if you are to have but one book with you upon a journey, let it be a book of science. When you have read through a book of entertainment, you know it, and it can do no more for you; but a book of science is inexhaustible'". [Per Mr. C. Fox.]

# THE TEACHING OF DIFFERENTIALS.

## I.

In the July number of the *Gazette* * Mr. E. G. Phillips says: "Frequently it happens that students whose early training has been on the lines of many of the existing elementary text-books on the Differential Calculus come up to the University never even having heard of a *differential*! It would be of the greatest possible assistance to those responsible for the later teaching of the Calculus if the schoolmasters taught the subject from the differential standpoint right from the start". I have the very greatest hesitation in subscribing to this latter statement.

In the case of a function of a single variable there is no distinction, logical or other, between the "derivative" and the "differential coefficient". The definition of the differential coefficient is that it is $A$ in the equation †

$$\Delta y = A\Delta x + \epsilon\Delta x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

provided that $A$ is independent of $\Delta x$ and $\epsilon \to 0$ as $\Delta x \to 0$. But to say that $\epsilon \to 0$ as $\Delta x \to 0$ in (2), *as it stands*, is meaningless; all we can deduce is that, if $A$ is finite, and if $\epsilon$ is bounded, then $\Delta y \to 0$ as $\Delta x \to 0$; in other words, under the conditions stated (2) implies the continuity of the function $y$ at the point $x$. Of course what is intended is that we shall divide (2) by $\Delta x$, and that if

$$\frac{\Delta y}{\Delta x} = A + \epsilon,$$

then $\epsilon \to 0$ as $\Delta x \to 0$; but this is only another way of saying that

$$A = \lim \frac{\Delta y}{\Delta x},$$

which is accordingly the definition of $A$; in other words, $A$ is logically the same as the derivative.

There remains the question of notation. If $y = f(x)$, and $dy$ and $dx$ are differentials, then $dx$ is an arbitrary number associated with the particular value $x$ of the independent variable, and $dy$ is defined by the equation

$$dy = f'(x)\, dx.$$

Of what use is this notation to the beginner? To me it seems that it cannot fail to be mystifying to an unsophisticated youth. For example, although the notation suggests the true relation

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

the suggestion is actually false, for the $dx$ in $dy/dx$ and the $dx$ in $dx/dt$ bear no relation whatever to one another; the first is arbitrary, the second is a definite multiple of the arbitrary number $dt$.

Mr. Phillips says ‡ that, without defining differentials, it is not

* Vol. XV. p. 401.

† Equation (2) of Phillips' paper.

‡ *Ibid.* p. 403.

allowable even to write $ds^2 = dx^2 + dy^2$, or $dy = g(x)\,dx$. This is, of course, true. The answer is that it is unnecessary and undesirable in elementary work to use either of these expressions; we may write, for the first, $s'^2 = x'^2 + y'^2$, or

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2,$$

if we wish to show the independent variable explicitly. As for $dy = g(x)\,dx$, it is most frequently written in course of changing the variable, or "substituting", in an integral, and it is then merely a careless way of writing $dy/dx = g(x)$. We have

$$\int f(y)\,dy = \int f(y)\,(dy/dx)\,dx = \int f(y)\,g(x)\,dx.$$

If, as is often done, we write

$$dy = g(x)\,dx, \quad \int f(y)\,dy = \int f(y)\,g(x)\,dx,$$

then, whatever meaning may be attached to $dy$ and $dx$ (if it is the same in the two equations), it is certain that they are not differentials; for to say that, in $\int F(x)\,dx$, $dx$ is an arbitrary number associated with the particular value $x$ of the independent variable, is nonsense. In fact $dx$, when separated from the integral sign, is meaningless.

I agree with Mr. Phillips that the notation $dy/dx$ should not be used to denote the derivative; indeed, the symbol is a very inconvenient one for the purpose. I do not know whether the intelligence of Adelaide undergraduates is much below the average, but, in obedience to the instruction "Write down the equation of the tangent to the parabola $y = x^2$ at the point $(1, 1)$", fifty per cent. of a class beginning the Calculus, having looked up the formula in a text-book (they don't trust their lecture-notes), will write, "The tangent is $y - 1 = (x - 1)\,dy/dx = 2x\,(x - 1)$",

an absurdity which becomes impossible, except to the most perverted ingenuity, when a reasonable notation is employed.

There is still one matter, of fundamental importance, on which I have to express an opinion at variance with that of Mr. Phillips. He writes *: "The *derivative* is defined as follows. Let $y = f(x)$ be a finite single-valued function of $x$ in a given interval $(a, b)$, and let $x$ be any point in this interval. If $x$ be given a (positive or negative) increment $\Delta x = h$, then the corresponding increment of $y$, which we denote by $\Delta y$, is $f(x + h) - f(x)$. The ratio of these increments is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If this ratio tends to a definite limit as $h$ tends to zero, this limit is called the *derivative* of $f(x)$ at the point $x$, and it is usually denoted by $f'(x)$".

* P. 401.

The wording has a familiar sound, but it none the less betrays confusion of thought. The confusion is due to the use of $x$ with two different meanings ; it is the independent variable, and corresponding to this meaning $y$ is the dependent variable ; it is also a fixed number, a value of the independent variable, and corresponding to this meaning $y$ is the corresponding value of the dependent variable. The word "increment" has reference to the second meaning of $x$ and $y$, and is, to say the least of it, inappropriate. The inherent absurdity of the definition is revealed by writing it out for a particular case :

"Let $y = \sin x$, and let $x = 0$. If 0 be given an increment $\Delta 0$, then the corresponding increment of $\sin 0\ (=0)$, which we denote by (what ?—) $\Delta 0$, is $\sin \Delta 0$." And so on.

The correct definition of the derivative is so simple as to afford no excuse for confused thought :

"Let the function $f(x)$ be defined in a neighbourhood of the point $x = a$. Then the derivative of $f(x)$ at $x = a$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided that this limit exists."

One word more, and I have done. If $f(x)$ is defined in an interval $(a, b)$, and if the derivative of $f(x)$ exists at every point of $(a, b)$, then the set of values of the derivative defines a function $f'(x)$ in $(a, b)$, the *derived function* of $f(x)$. The derived function is logically distinct from the derivative, and I would keep myself the notation $dy/dx\ [= f'(x)]$ for the derived function of $y\ [= f(x)]$. J. R. W.

## II.

The gratitude one feels to Mr. Phillips for making so admirably clear the difference between a derivative and a differential coefficient is clouded with a doubt as to whether every University teacher would wish his students to have heard of differentials at school. Some of us find so strong a tendency to regard the derivative as only a first approximation, instead of a limit, that we have banished differentials from our Intermediate Course and endeavour to prevent our students from hearing of them until their second year. It must be remembered that we can assume no knowledge of the Calculus at all in our Intermediate students, and that only a small proportion of them are going on afterwards to Honours Mathematics.

However horrifying this may seem, I cannot help thinking that this censorship might be continued, and Mr. Phillips' trouble met, by avoiding the use of the term "differential coefficient" and—if he wishes—the use of the symbol $dy/dx$, until students are sufficiently accustomed to finding derivatives for them to be introduced to differentials with safety. In any case, I believe very firmly that, *however good the teacher, it is not safe to give students any inkling of differentials until they are thoroughly accustomed to dealing with derivatives as exact limits.*

As the censorship just proposed is rather drastic, I may perhaps outline a course which is less of a self-denying ordinance and is in effect a re-arrangement of that set out by Mr. Phillips. Starting with $y=f(x)$, we can introduce $\Delta x$ and $\Delta y$ as increments and assume that $\Delta y$ can be written in the form

$$\Delta y = \phi(x) \,.\, \Delta x + \epsilon \,.\, \Delta x,$$

where $\phi(x)$ does not contain $\Delta x$ and $\epsilon \to 0$ as $\Delta x \to 0$. Then $y$ is said to be differentiable with respect to $x$ and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \phi(x).$$

We write this limit as $dy/dx$ and call $\phi(x)$ the differential coefficient of $y$ with respect to $x$.

In this way we come easily to the idea of the differential coefficient from the idea of graphs already familiar to the student, and can illustrate the limiting process by comparison with chord and tangent.

Next we introduce the (to the student) harder idea of the derivative by pointing out that the differential coefficient depends only on the function $f(x)$, defining the derivative as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

when this limit exists.

At a later stage—and almost as late as we please—we can introduce the differential $dy$ and point out that what we have by anticipation called the differential coefficient is in fact the coefficient of the differential $dx$. In this way students do not catch the approximation fever because they have been previously inoculated with limits.

To avoid misunderstanding, I should like in conclusion to state quite plainly that the only part of Mr. Phillips' paper to which I am taking exception is the single sentence : " It would be of the greatest possible assistance to those who are responsible for the later teaching of the Calculus if the schoolmasters taught the subject from the differential standpoint right from the start ". And I hope I have made it clear that my objection is in no way based on any idea that the schools would not teach the subject from that standpoint as ably as Mr. Phillips has expounded it.

J. T. COMBRIDGE.

## III.

The opportunity has been given to me of replying to two criticisms of my article on " The Teaching of Differentials ".

Both the writers question the wisdom of my suggestion " that it would be of the greatest possible assistance . . . if the schoolmasters taught the subject from the differential standpoint right from the start". This was merely an expression of my own opinion, and although I recognise that some difficulties might arise from it, in

substance my opinion remains unchanged. No further remark on Mr. Combridge's very fair and constructive criticisms is necessary. The reply of Mr. Wilton seems to me to call for several remarks.

(1) I maintain that the debatable question as to whether " in the case of functions of a single variable there is any distinction (logical or other) between the ' derivative ' and the ' differential coefficient ' " is trivial as compared with the importance of keeping the two concepts distinct at the initial stage : because, in the case of functions of more than one variable, this distinction is fundamental to the treatment of the subject from the differential standpoint.

(2) I state that $$\Delta y = A\Delta x + \epsilon\Delta x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$ provided that $A$ is independent of $\Delta x$ and $\epsilon \to 0$ as $\Delta x \to 0$.

I do not agree with Mr. Wilton's remark that " to say that $\epsilon \to 0$ as $\Delta x \to 0$ in (2) *as it stands* (the italics are his) is meaningless ". It is not meaningless to me, and it is a generally accepted form of statement in modern analysis.*

(3) Mr. Wilton states that it is unnecessary and undesirable to write $ds^2 = dx^2 + dy^2$ and $dy = g(x)\,dx$ in elementary work. If differentials have been *properly defined* I do not agree that it is " undesirable ". I do, however, object to the common practice of using these equations involving differentials when $dy/dx$ has *only* been defined as $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

I do not consider that Mr. Wilton's statement " $dy = g(x)\,dx$ is merely a careless way of writing $dy/dx = g(x)$ " can ever justify its use, above all by beginners in the subject.

(4) The notation $\int F(x)\,dx$ for an *indefinite* integral has the sanction of custom, but it is open to criticism. I have suggested elsewhere † that the notation $D^{-1}F(x)$ might in some respects be better. The adoption of some other notation for an indefinite integral would render unnecessary Mr. Wilton's statement that $dx$ when it occurs separated from the integral sign is meaningless. What Mr. Wilton is there criticising is in reality the unsatisfactory notation commonly used for an indefinite integral.

(5) I do not agree that my definition of the derivative " betrays confusion of thought ". Although $x$ has been used with two different meanings, as Mr. Wilton points out, I have never yet known this practice (which, by the way, is quite common in many parts of Mathematics) to lead to any difficulty in understanding the definition. In any case I should assume that anyone who had read my article would at once substitute $a$ for $x$ when it stands for the " fixed number " if he really thought that it would be an advantage when actually teaching the subject. I may remark that if I err in respect of this definition I do so in good company, for my definition is

* See de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, i (1921), 51.

† *A Course of Analysis* (Cambridge 1930), 165 (f.n.).

exactly the same as that given by de la Vallée Poussin,* whose clarity of exposition and freedom from "confusion of thought" would, I think, be generally admitted.

In conclusion, I wish to state that, having availed myself of this opportunity of replying to criticisms, so far as I am concerned, the matter is definitely closed.

E. G. PHILLIPS.

IV.

Mr. Phillips prefaces his article with a very brief historical note: it may be of interest, however, to point out that the differential *did* make an appearance in the Calculus text-books in this country during the nineteenth century,† usually in a vague and inaccurate manner. But towards the end of the century an attempt was made, particularly in Cambridge,‡ to introduce into England the results of the brilliant progress of the Theory of Functions on the Continent. This movement, bringing with it Weierstrass' non-differentiable function and similar warnings, created a distrust of loose arguments about "quantities", "infinitesimals" and "orders of magnitude". Thus in the first years of this century, Cambridge teaching of Analysis had banned the differential; it is nowhere mentioned in the first edition of Professor G. H. Hardy's *Pure Mathematics* (1908), while in the second edition (1914) a paragraph is devoted to a rather uninteresting discussion of the differential of a function of a single variable, nothing being said about functions of two or more variables, although this is precisely the domain in which the difference between differentials and derivatives is made clear and valuable. It is possible that the first satisfactory account of the differential in English is that to be found in Professor W. H. Young's Cambridge Tract, *The Fundamental Theorems of the Differential Calculus* (1910), but by the end of the War the rigorous exposition of the differential had become part of the stock-in-trade of the Cambridge teachers.

T. A. A. B.

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**833.** Another biographer tells us that Cromwell "excelled chiefly in the Mathematics" . . . [Cromwell writes to his son Richard] "Read a little history . . . study the Mathematics and cosmography. These are good with subordination to the things of God. These fit for public services for which a man is born".—C. H. Firth, *Oliver Cromwell*, p. 6 (1900). [Per Mr. J. W. Stewart.]

**834.** "Ah! now you have a fine fish. . . . For this stream [the Mole] it is a very excellent fish, exactly three pounds weight, I find. How do I know it is just three pounds? I will tell you." He knows because he has measured the fish and finds him nineteen inches long by ten in girth, and if you do the sum his way, it works out at three pounds. "This is in accordance, as you suppose, with the mathematical law that similar solids are to each other in the triplicate ratio of one of their dimensions." That is the way to measure trout in Norbury Park.—C. J. Swete in his *Handbook of Epsom*, 1853. See Eric Parker, *Highways and Byways in Surrey*. [Per Miss M. M. Haslam.]

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* *Loc. cit.* 49.

† *E.g.* B. Williamson, *Differential Calculus*, pp. 3-5 (1899)

‡ Notably Prof. A. R. Forsyth's *Theory of Functions* (1893).

## TO BORDER A SQUARE OF THE 5TH ORDER—CONSECUTIVE NUMBERS.

By J. C. Burnett.

*Explanation of terms used.* For "Bordered Squares" the term *Heart* is given to the core around which the border is to be arranged. The Heart can be one of several different kinds. In our case, the rows and columns do not sum to the same amount; but the Heart is *Associated*, that is, every pair of numbers equidistant from the centre of the Heart in a straight line sums to the same amount, here 50. The numbers which make up such a pair are *Complementary Numbers*. When the numbers of these pairs are subtracted one from the other, they give the *Complementary Differences*.

The square we consider is an associated Heart as regards the complementary numbers, but the rows and columns require the numbers in heavy type placed against them to bring them up to the magic number 175 for the rows and columns of a square of the 7th order.

| | | | | | |
|---|---|---|---|---|---|
| 46 | 18 | 3 | 16 | 43 | **49** |
| 28 | 33 | 41 | 10 | 27 | **36** |
| 39 | 12 | 25 | 38 | 11 | |
| 23 | 40 | 9 | 17 | 22 | **64** |
| 7 | 34 | 47 | 32 | 4 | **51** |
| **32** | **38** | | **62** | **68** | |

In order 7, the complementary differences are all the even numbers up to 48, so that half-differences can be taken as the first 24 integers. There are 12 of these 24 differences in the Heart, leaving 12 differences over, out of which 8 may be taken to compose the numbers in heavy type, leaving 4 complete complementary pairs for the corner numbers and for the medianes, or central row and column.

One arrangement is as follows, but there are several others.

| *A.* | | *B.* | |
|---|---|---|---|
| 49 = 44 + 5 | | 68 = 49 + 19 | |
| 36 = 21 + 15 | | 62 = 48 + 14 | |
| 64 = 29 + 35 | | 38 = 36 + 2 | |
| 51 = 45 + 6 | | 32 = 31 + 1 | |

This only leaves the following four differences:

| | | | |
|---|---|---|---|
| 42 − 8 = 34 : | 17 | 30 − 20 = 10 : | 5 |
| 37 − 13 = 24 : | 12 | 26 − 24 = 2 : | 1 |

Keeping the two sets of numbers marked *A* and *B* separate, the possible differences must be calculated. Each set can have eight, and these are now given in full:

| | | | | | Total. | | | | | Total. | Diff. | ½-Diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *A.* | 44 | 21 | 29 | 45 | 139 | 5 | 15 | 35 | 6 | 61 | 78 | **39** |
| | 44 | 21 | 29 | 6 | 100 | 5 | 15 | 35 | 45 | 100 | 0 | **0** |
| | 44 | 21 | 35 | 45 | 145 | 5 | 15 | 29 | 6 | 55 | 90 | **45** |
| | 44 | 21 | 35 | 6 | 106 | 5 | 15 | 29 | 45 | 94 | 12 | **6** |
| | 44 | 15 | 29 | 45 | 133 | 5 | 21 | 35 | 6 | 67 | 66 | **33** |
| | 44 | 15 | 29 | 6 | 94 | 5 | 21 | 35 | 45 | 106 | 12 | **6** |
| | 44 | 15 | 35 | 45 | 139 | 5 | 21 | 29 | 6 | 61 | 78 | **39** |
| | 44 | 15 | 35 | 6 | 100 | 5 | 21 | 29 | 45 | 100 | 0 | **0** |
| *B.* | 49 | 48 | 36 | 31 | 164 | 19 | 14 | 2 | 1 | 36 | 128 | **64** |
| | 49 | 48 | 36 | 1 | 134 | 19 | 14 | 2 | 31 | 66 | 68 | **34** |
| | 49 | 48 | 2 | 31 | 130 | 19 | 14 | 36 | 1 | 70 | 60 | **30** |
| | 49 | 48 | 2 | 1 | 100 | 19 | 14 | 36 | 31 | 100 | 0 | **0** |
| | 49 | 14 | 36 | 31 | 130 | 19 | 48 | 2 | 1 | 70 | 60 | **30** |
| | 49 | 14 | 36 | 1 | 100 | 19 | 48 | 2 | 31 | 100 | 0 | **0** |
| | 49 | 14 | 2 | 31 | 96 | 19 | 48 | 36 | 1 | 104 | 8 | **4** |
| | 49 | 14 | 2 | 1 | 66 | 19 | 48 | 36 | 31 | 134 | 68 | **34** |

Now, using the four remaining differences, we apply the rule for complementary differences when bordering a Heart: "That no two differences which appear on the same side of an equality sign in the first equation shall so appear in the second equation, and similarly for any two differences which appear on opposite sides of an equality sign". There are several here given:

(i) $12+5=17+(\mathbf{0})$, $\quad 12=5+1+(\mathbf{6})$

(ii) $12+1+(\mathbf{4})=17$, $\quad 5+1+(\mathbf{6})=12$

(iii) $17+12+5=0+(\mathbf{34})$, $\quad 5+1+(\mathbf{6})=12$

(iv) $17+12+1=0+(\mathbf{30})$, $\quad 5+1+(\mathbf{6})=12$

(v) $17+12+1=0+(\mathbf{30})$, $\quad 12+5+(\mathbf{0})=17$

The border can now be constructed by translating one of these equations into its corresponding large and small numbers and inserting the proper series of numbers for the difference. A repeated difference gives another border.

The complete bordered square for (i) is:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **37** | **1** | **2** | 8 | **48** | **49** | 30 | |
| **5** | 46 | 18 | 3 | 16 | 43 | **44** | |
| **15** | 28 | 33 | 41 | 10 | 27 | **21** | |
| 24 | 39 | 12 | 25 | 38 | 11 | 26 | Magic Number 175. |
| **29** | 23 | 40 | 9 | 17 | 22 | **35** | |
| **45** | 7 | 34 | 47 | 32 | 4 | **6** | |
| 20 | **31** | **36** | 42 | **14** | **19** | 13 | |

Another completely different set of numbers composing the numbers required to bring the rows and columns of the Heart up to 175 is :

| | |
|---|---|
| $49 = 48 + 1$ | $68 = 37 + 31$ |
| $36 = 21 + 15$ | $62 = 42 + 20$ |
| $64 = 35 + 29$ | $38 = 30 + 8$ |
| $51 = 49 + 2$ | $32 = 19 + 13$ |

leaving

| | |
|---|---|
| $45 - 5 = 40 : \underline{20}$ | $36 - 14 = 22 : \underline{11}$ |
| $44 - 6 = 38 : \underline{19}$ | $26 - 24 = 2 : \underline{1}$ |
| $19 + 1 = 20 + (0)$ | $20 + 19 = 11 + (28).$ |

One of the differences is 0 and the other is 28 ; the border is

$$\begin{Bmatrix} 45 & 44 & 13 & 8 & 14 & 20 & 31 \\ 6 & 5 & 19 & 30 & 36 & 42 & 37 \end{Bmatrix}$$

$$\begin{Bmatrix} 45 & 6 & 1 & 15 & 24 & 35 & 49 \\ 44 & 5 & 48 & 21 & 26 & 29 & 2 \end{Bmatrix}$$

J. C. BURNETT.

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**835.** In the dusk of a lane I met a shepherd with his sheep. A small dog with the expression of a professor of mathematics does all the work.—H. V. Morton, *In Search of Scotland*, p. 8. [Per Prof. J. P. Dalton.]

**836.** Londonderry : printed for the Company of Stationers by A. M. C., Professor of Prediction by Stargazing.—Imprint of the *Belfast Annual Pocket-Book* for 1819. [Per Prof. E. H. Neville.]

**837.** From a script. "That is, there are two straight lines through $P$ parallel to $AB$, which is impossible by Fairplay." [Per Mr. F. C. Boon.]

**838.** Sophie Charlotte is she who wrote once "Leibnitz talked to me of the infinitely little (*de l'infiniment petit*) : *mon Dieu*, as if I did not know enough of that ! "—Carlyle, *History of Frederick the Great*, i, 37 (1872).

[Sophie Charlotte, of Hanover ; sister of George I ; wife of Frederick, 12th Elector of Brandenburg and first King in Prussia ; grandmother of Frederick the Great ; a great admirer of Leibnitz and his philosophy.]

**839.** We can never regard that as a scientific interpretation of Nature which applies mathematical processes or laws to the behaviour of bodies in space, but forgets the mind that compels man to think the pure ideas of his reason.—Fairbairn, *The Philosophy of the Christian Religion*, p. 56. [Per Mr. J. W. Stewart.]

**840.** The mathematics which have controlled and guided the Builder of the heavens, are identical with the mathematics which the astronomer in his study deduces from the idea of space given in his own thoughts, and which he proves by the processes of his own reason.—*Ibid.* p. 37. [Per Mr. J. W. Stewart.]

**841.** By the end of 1905, the tension was such that nothing could long avert a fratricidal struggle on a gigantic scale, except some great melting of hearts, some wave of inspiration which should lift the dull, deadly antagonisms of the hour to a level so high that—even as a mathematical quantity passing through infinity changes its sign—they would become actual unities.—Winston S. Churchill, *If Lee had won the battle of Gettysburg*, p. 189, from *If it had happened otherwise*, edited by J. C. Squire, 1931. [Per Mr. D. E. Collier.]

# INTERPOLATION WITHOUT PRINTED DIFFERENCES.

## JORDAN'S AND AITKEN'S FORMULAE.

I.* By J. Wishart, M.A., D.Sc.

I do not know what proportion of the users of mathematical tables go beyond the use of proportional parts in interpolation work. But there are undoubtedly, on the one hand, some tables whose construction would be unnecessarily laborious and expensive were they to be at such an argument interval as to render linear interpolation feasible, and, on the other hand, there are those who find it necessary to use such tables. Given such a table, there are various methods of performing the interpolation.† There is the use of the Taylor series, $f(a+h)=f(a)+hf'(a)+\frac{1}{2}h^2f''(a)+\dots$, which requires a knowledge of the derivatives of the function as well as the function itself. Then there is the use of the Lagrange formula of interpolation, $f_x=a_0f_0+a_1f_1+a_2f_2+\dots$, involving a number of tabular values on both sides of the required value.

Lastly, there is the use of one or other of the interpolation formulae involving the differences of the tabulated function, for example, Gauss, Newton-Stirling, Newton-Bessel, Everett. As an example of this type I will write down the last of these, due to Everett:

$$u_x=\phi u_0+\theta u_1-\frac{\theta\phi}{3!}[(1+\phi)\delta^2u_0+(1+\theta)\delta^2u_1]$$
$$+\frac{\theta\phi(1+\theta)(1+\phi)}{5!}[(2+\phi)\delta^4u_0+(2+\theta)\delta^4u_1]-\dots,$$

$0<\theta<1$, $\phi+\theta=1$, $\theta$ being the fraction of the argument interval from $u_0$ to the value for which it is desired to interpolate.

It is always the business of the table maker to arrange his argument interval in such a way that interpolation is possible, that is, the table is not to be regarded as giving a series of isolated values of a function, but should be capable of giving any value whatever throughout the entire range. The researches of Steffensen ‡ have shown that the size of the remainder term should always be considered. Apart from this the table maker may economise in his volume at the cost of making interpolation difficult for the user, for if he use a wide argument interval, subject, of course, to what has been said about the remainder term, the user will have to go up to high orders of differences to secure full accuracy. On the other hand, he has the choice of amplifying his table in such a way as to make interpolation feasible with only a small number of differences. Theoretically, then, the table is complete when so prepared, and the

* A paper read at the Centenary Meeting of the British Association, Sept. 1931.

† For further details, see W. F. Sheppard, *Gazette*, XV, 1930, 232-249.

‡ *Interpolation*, 1927 (The Williams and Wilkins Co., Baltimore).

user has to do two things: (1) find his differences up to the order required, and (2) apply his formula. A desire, however, to lighten the task of the interpolater has led to a general introduction of printed differences with the table itself. For the first difference this practice is as old as the earliest tables of logarithms, for further orders of differences it is more recent. We now see why Everett's formula has attained its popularity, for it only requires the even differences of two entries adjacent to the value required, and thus saves a good deal of the time taken in calculating all differences (to obtain $\delta^2 u$ by machine calculation is one operation, and is quicker than obtaining two first differences), and rather more than half the printing of them.

Everett's formula has now been provided with three tables of the coefficients,* so that all the user has to do is to perform a series of multiplications of difference and coefficient, and add or subtract the results—an operation the whole of which is performed continuously on the calculating machine. The table maker may then economise on the production of his table to the extent of providing a table of such an argument interval as to necessitate the printing of 2nd, 4th, and sometimes 6th differences, without giving the user an unnecessary amount of labour. To go beyond this and require 8th differences is hardly advisable, particularly as coefficients of the 8th differences are not available, although something can be done, as illustrated in the volume prepared by the Committee for the Calculation of Mathematical Tables,† to throw the 8th differences back on to the 6th, provided they are not too large.

If we want further economies in the production of tables, then we must modify our procedure. In giving as the title of this paper "Interpolation without Printed Differences", I wish to draw attention to a recent new formula of C. Jordan ‡ in which the previous preparation and tabulation of differences is of no advantage whatever to the user. The formula was derived from Everett and has since been derived by Aitken from the Newton-Bessel formula and by Lidstone from general principles §; it is as follows: perform a series of linear interpolations between certain of the tabular values as shown in the following scheme:

| | |
|---|---|
| between $u_0$ and $u_1$ | $\phi u_0 + \theta u_1$ |
| between $u_{-1}$ and $u_2$ | $\frac{1}{3}\{(1+\phi)u_{-1} + (1+\theta)u_2\}$ |
| between $u_{-2}$ and $u_3$ | $\frac{1}{3}\{(2+\phi)u_{-2} + (2+\theta)u_3\}$ |
| ... | ... |

where $0 < \theta < 1$, and $\phi + \theta = 1$. The later stages in the calculation will show how far one should go; but this might also be conveniently

* A. J. Thompson, *Tracts for Computers*, No. v (Cambridge); E. Chappell, *Interpolation Coefficients*, 1929 (The author; 41 Westcomb Park Road, S.E. 3); J. W. Glover, *Tables of Applied Mathematics in Statistics*, 1924 (Wahr).

† *Mathematical Tables*, Vol. I, 1931. (British Association.)

‡ *Metron*, vii, 3 (1928), 47-51.

§ See the papers following this one, pp. 18, 22.

indicated on each page of the table, which, be it remembered, will not have the differences printed. If these linear interpolates are represented by $a, b, c, d, e, \ldots$, the final interpolate is

$$a - G^{II}(b-a) + G^{IV}(c-3b+2a) - G^{VI}(d-5c+9b-5a) + G^{VIII}(e-7d+20c-28b+14a)\ldots.$$

The easiest way to form the functions in brackets above is to write down the symmetrical series

| | | | | |
|---|---|---|---|---|
| $d$ | | | | |
| $c$ | | | | |
| $b$ | | | | |
| $a$ | | | | |
| $a$ | $b-a$ | $c-3b+2a$ | $d-5c+9b-5a$ | |
| $b$ | $c+a-2b$ | $d-4c+6b-3a$ | ... | |
| $c$ | $d+b-2c$ | ... | | |
| $d$ | ... | | | |
| ⋮ | | | | |

and calculate the even differences on one side of the central line (note for the first line that the first difference of the functions in the preceding column is all that is required, so that the part above the line is not really required in the calculation, but is here introduced to show the way the functions are built up). It is this stage which shows us how far to proceed with the calculation of the linear interpolates, for when the bracketed function falls below a certain value it will be negligible when taken in conjunction with the appropriate $G$ coefficient. $G^{II}$, $G^{IV}$, etc., are the even coefficients, taken with positive sign, of the Gauss interpolation formula,* and, as it happens, are available in tabulated form up to $G^{VI}$ for every ·001 of the argument interval in Chappell's tables of Interpolation Coefficients. No doubt further tables of these coefficients will be forthcoming if need is felt for them; in fact Jordan stated in 1928 that a table along these lines was in course of preparation. It will be seen that there are three stages in the operation of applying the formula:

(1) a series of linear interpolates;

(2) their differencing;

(3) a series of multiplications with known coefficients similar to the Everett calculation but with about half the number of operations (working with Everett up to $\delta^6$ is equivalent to going up to $G^{VI}$ with the Jordan formula).

Now, it is not claimed that an interpolate is obtained as rapidly with the Jordan formula as with the use of Everett's formula with the differences already provided in the table, but it is certainly appreciably faster than the combined operations of (1) differencing

* $G^{II} = \theta(1-\theta)/2!$; $G^{IV} = \theta(1-\theta^2)(2-\theta)/4!$; $G^{VI} = \theta(1-\theta^2)(4-\theta^2)(3-\theta)/6!$, etc.

the appropriate part of a table not already provided with the differences, and (2) using Everett's formula. The extent of the difference in time in this comparison seems to vary with the skill of the computer and whether or not he has a calculating machine available. In this connection the type of machine will also make a difference. Whatever the position will ultimately be found to be, however, intermediate between the cases of using Everett's formula (1) with differences provided, and (2) with no differences given, it seems clear that the new formula does something to bridge the gap, and certainly, if its use became general, it would lead to a great economy in table construction, as differences would no longer require to be printed. Theoretically, of course, it is not necessary to print the differences, as they can be calculated from the table, but it has generally been held to be advisable to provide these to lessen the labour of interpolation to the user. If the adoption of the Jordan formula does not increase disproportionately the work required from the user—does not, in fact, lead us back to where we might be with other formulae and without differences—then it has a great deal to commend it. The point has been considered by the Committee for the Calculation of Mathematical Tables; and while it has decided that in the volume just published,* consisting all of single-entry tables, the expense of printing the necessary number of even differences was justified, it is open to consider whether in its future programme, which will involve consideration of double- and even triple-entry tables, where the difficulty (and expense) of presenting all the necessary differences will be very much greater, it will not be advisable to dispense with these and recommend the wholesale adoption of the new method.

The Jordan formula is certainly a very interesting one, and deserves to take an honoured place beside those others associated by their names with some of the greatest of mathematicians. It has already attracted the attention of some mathematicians in this country; Aitken † deduces the formula from the Newton-Bessel formula, and then noting that this interpolates by means of an even number of tabular entries, proceeds to deduce from the Newton-Stirling formula, which interpolates by means of an odd number of entries, a new formula analogous to Jordan's which involves linear interpolation on first differences of the function, not on the functional values themselves. This is reminiscent of the second Everett formula, deduced by Lidstone ‡ for an odd number of entries, and it is significant that Lidstone himself, whose services in interpolation work are widely known and appreciated much beyond the circle of actuaries for whom the work was done, has also interested himself in the Jordan formula, and in his note he exhibits the rationale of the formula in a very interesting way.§

J. WISHART.

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* British Association, *loc. cit.* † See p. 18.

‡ *Journal of the Institute of Actuaries*, lx. (1929), 349-352.

§ See p. 22.

## II. By A. C. Aitken, D.Sc.

1. The novel features of Jordan's formula have been sufficiently described in Dr. Wishart's paper. We shall content ourselves, therefore, with deriving the formula, as well as another formula of the same kind, but in some respects significantly different.

Jordan deduced his formula by transforming Everett's well-known double-line central difference formula. If he had similarly transformed the less widely known "second Everett" formula, he would have obtained the second formula of Jordan type which we derive below. Our own course will be to transform the Newton-Bessel formula into Jordan's, and the Newton-Stirling into the companion to Jordan's, by means of the relations, not uninteresting in themselves, which exist between the operations of averaging and central differencing at various ranges. We feel, however, that Dr. Lidstone's demonstration, given after this, more truly puts in evidence the theoretical basis of these formulae.

2. The relations referred to above depend on simple algebraic identities. Thus we have

$$\frac{1}{2n+1}(t-t^{-1})^{2n+1}=\frac{1}{2n+1}(t^{2n+1}-t^{-2n-1})$$
$$-\frac{c_1}{2n-1}(t^{2n-1}-t^{-2n+1})+\frac{c_2}{2n-3}(t^{2n-3}-t^{-2n+3})-\dots, \quad \dots(1)$$

where
$$c_r=\binom{2n+1}{r}\frac{2n-2r+1}{2n+1},$$

and, by differentiating this once,

$$(t+t^{-1})(t-t^{-1})^{2n}=(t^{2n+1}+t^{-2n-1})-c_1(t^{2n-1}+t^{-2n+1})$$
$$+c_2(t^{2n-3}+t^{-2n+3})-\dots\dots\dots\dots(2)$$

We next define the operation of "central averaging at range $m$" by the relation
$$\mu_m f_0=\tfrac{1}{2}\{f_{\frac{1}{2}m}+f_{-\frac{1}{2}m}\}, \quad \dots\dots\dots\dots(3)$$

so that $\mu_1$ is Sheppard's $\mu$; and of "central differencing at range $m$" by
$$\delta_m f_0=f_{\frac{1}{2}m}-f_{-\frac{1}{2}m}. \quad \dots\dots\dots\dots(4)$$

Then if, in the usual symbolic notation of Finite Differences, $Ef_x=f_{x+1}$, we have
$$\delta_m=E^{\frac{1}{2}m}-E^{-\frac{1}{2}m}, \quad \dots\dots\dots\dots(5)$$
$$\mu_m=\tfrac{1}{2}[E^{\frac{1}{2}m}+E^{-\frac{1}{2}m}].$$

Substituting $E^{\frac{1}{2}}$ for $t$ in (1) and (2) we have the required relations,

$$\frac{1}{2n+1}\delta^{2n+1}=\frac{1}{2n+1}\delta_{2n+1}-\frac{c_1}{2n-1}\delta_{2n-1}+\frac{c_2}{2n-3}\delta_{2n-3}-\dots, \quad \dots\dots(6)$$
$$\mu\delta^{2n}=\mu_{2n+1}-c_1\mu_{2n-1}+c_2\mu_{2n-3}-\dots.$$

Let us now take the Newton-Bessel formula and group the terms in bracketed pairs, as follows :

$$f_{x-\frac{1}{2}} = \{\mu f_0 + x\delta f_0\} + \frac{x^2-\frac{1}{4}}{2!}\left\{\mu\delta^2 f_0 + \frac{x}{3}\delta^3 f_0\right\} + \frac{(x^2-\frac{1}{4})(x^2-\frac{9}{4})}{4!}\left\{\mu\delta^4 f_0 + \frac{x}{5}\delta^5 f_0\right\} + \dots \dots \dots\dots(7)$$

Substituting from (6) we have for a typical bracket

$$\mu\delta^{2r} f_0 + \frac{x}{2r+1}\delta^{2r+1} f_0 = \left\{\mu_{2r+1} + \frac{x}{2r+1}\delta_{2r+1}\right\} f_0 - c_1\left\{\mu_{2r-1} + \frac{x}{2r-1}\delta_{2r-1}\right\} f_0 + \dots \dots \dots\dots(8)$$

Each pair of terms on the right of (8) resembles the first pair of terms in the Bessel formula itself, and so constitutes a linear interpolation at a certain range : the first pair, for example, is for $f_{x\ \frac{1}{2}}$, as found by proportional parts from $f_{\frac{1}{2}(2r+1)}$ and $f_{-\frac{1}{2}(2r+1)}$. If we denote the results of these symmetrical linear interpolations at ranges 1, 3, 5, ... by $g_1$, $g_3$, $g_5$, ... , the Newton-Bessel formula takes the shape

$$f_{x-\frac{1}{2}} = g_1 + \frac{x^2-\frac{1}{4}}{2!}(g_3 - g_1) + \frac{(x^2-\frac{1}{4})(x^2-\frac{9}{4})}{4!}(g_5 - 3g_3 + 2g_1) + \dots , \quad \dots(9)$$

and this is Jordan's formula. The coefficients of the $g$'s in any bracket are those of positive (or negative) odd powers of $t$ in the expansion of

$$(t+t^{-1})(t-t^{-1})^{2r}, \quad r = 0, 1, 2, \dots \dots \dots\dots\dots\dots(10)$$

It follows from (10) that the linear compounds

$$g_3 - g_1,\ g_5 - 3g_3 + 2g_1, \dots ,$$

are really *mean central differences*, for the first factor, $(t+t^{-1})$, implies symbolically the operation of averaging, $\mu$, and the second implies a central differencing operation, $\delta^{2r}$. Ocular demonstration of this fact can be given. Suppose, for example, that $n=2$. The function is then $g_5 - 3g_3 + 2g_1$, which we may write

$$\tfrac{1}{2}(g_5 - 3g_3 + 2g_1 + 2g_1 - 3g_3 + g_5).$$

From the table below, which shows $(t-t^{-1})^4$ multiplied by $\frac{1}{2}(t+t^{-1})$ by detached coefficients,

| | $g_5$ | $g_3$ | $g_1$ | $g_1$ | $g_3$ | $g_5$ |
|---|---|---|---|---|---|---|
| | 1 | −4 | 6 | −4 | 1 | |
| $\times\frac{1}{2}$ | | 1 | −4 | 6 | −4 | 1 |
| | 1 | −3 | 2 | 2 | −3 | 1 |

.....(11)

it is evident that $g_5 - 3g_3 + 2g_1$ is the mean fourth central difference of $g_5$, $g_3$, $g_1$, $g_1$, $g_3$, $g_5$, and the general result is inferred similarly.

3. The Newton-Bessel formula, and Jordan's, interpolate by means of an *even* number of tabular values. When the number of

data is *odd*, the formula that corresponds to the Newton-Bessel is the Newton-Stirling. In transforming it analogously, we must observe that the algebraic identities have an important difference in sign, which entirely alters the relations between the operators. We begin from

$$\frac{1}{2n}(t-t^{-1})^{2n}=\frac{1}{2n}(t^{2n}+t^{-2n})-\frac{c_1'}{2n-2}(t^{2n-2}+t^{-2n+2})$$
$$+\frac{c_2'}{2n-4}(t^{2n-4}+t^{-2n+4})-\dots\,,\quad\dots(12)$$

where $$c_r'=\binom{2n}{r}\frac{2n-2r}{2n},$$

and obtain by differentiation

$$(t+t^{-1})(t-t^{-1})^{2n-1}=(t^{2n}-t^{-2n})$$
$$-c_1'(t^{2n-2}-t^{-2n+2})+c_2'(t^{2n-4}-t^{-2n+4})-\dots\dots\dots(13)$$

Putting $t=E^{\frac{1}{2}}$ in these, we have

$$\frac{1}{2n}\delta^{2n}=\frac{1}{2n}\mu_{2n}-\frac{c_1'}{2n-2}\mu_{2n-2}+\frac{c_2'}{2n-4}\mu_{2n-4}-\dots\,,\quad\dots\dots\dots(14)$$
$$\mu\delta^{2n-1}=\delta_{2n}-c_1'\delta_{2n-2}+c_2'\delta_{2n-4}-\dots\dots\dots\dots\dots$$

Now taking the Newton-Stirling formula with terms after the first grouped in bracketed pairs, thus,

$$f_x=f_0+x\left\{\mu\delta f_0+\frac{x}{2}\delta^2 f_0\right\}+\frac{x(x^2-1)}{3!}\left\{\mu\delta^3 f_0+\frac{x}{4}\delta^4 f_0\right\}+\dots\,,\quad\dots(15)$$

we may substitute from (14) and obtain for a typical bracket in (15)

$$\mu\delta^{2r-1}f_0+\frac{x}{2r}\delta^{2r}f_0=\left\{\delta_{2r}+\frac{x}{2r}\mu_{2r}\right\}f_0$$
$$-c_1'\left\{\delta_{2r-2}+\frac{x}{2r-2}\mu_{2r-2}\right\}f_0+\dots\dots\dots\dots(16)$$

If $\left\{\delta_{2r}+\frac{x}{2r}\mu_{2r}\right\}f_0$ is denoted by $h_{2r}$, the Newton-Stirling formula becomes

$$f_x=f_0+xh_2+\frac{x(x^2-1)}{3!}(h_4-2h_2)$$
$$+\frac{x(x^2-1)(x^2-4)}{5!}(h_6-4h_4+5h_2)+\dots\,,\quad\dots\dots(17)$$

and this is the desired companion formula to Jordan's. It may be noted that these $h$'s are a kind of linear interpolation on first differences of $f_x$ at widening ranges, and are easily calculated by

$$h_{2r}=\{(r+x)(f_r-f_0)+(r-x)(f_0-f_{-r})\}/2r$$
$$=\{(r+x)f_r-2xf_0-(r-x)f_{-r}\}/2r,\dots\dots\dots\dots\dots(18)$$

whereas the $g$'s of Jordan's formula are calculated by the slightly simpler formula

$$g_{2r+1} = \{(r+x)f_{\frac{1}{2}(2r+1)} + (r-x)f_{-\frac{1}{2}(2r+1)}\}/(2r+1), \quad \ldots\ldots\ldots(19)$$

in fact by ordinary proportional parts.

A table of detached coefficients, for example

| | $h_6$ | $h_4$ | $h_2$ | 0 | $h_2$ | $h_4$ | $h_6$ | |
|---|---|---|---|---|---|---|---|---|
| | 1 | −5 | 10 | −10 | 5 | −1 | | |
| $\times\frac{1}{2}$ | | 1 | −5 | 10 | −10 | 5 | −1 | , ……(20) |
| | 1 | −4 | 5 | 0 | −5 | 4 | −1 | |

displays the linear compounds of the $h$'s as the *mean central odd differences* of the sequence

$$h_{2r},\ h_{2r-2},\ \ldots,\ h_2,\ h_0 = 0,\ -h_2,\ -h_4,\ \ldots,\ -h_{2r}. \quad \ldots\ldots(21)$$

The interpolating polynomials used in Jordan's formula are the Gauss coefficients of even order ; those used in the second formula are the Gauss coefficients of odd order. Tables of Gauss coefficients are available.

4. The full benefit of formulae of Jordan type is only realized when a machine is at hand and no differences of the tabular values of the function are provided. It would seem, indeed, that whereas the more familiar difference formulae came into favour originally because they avoided the heavy multiplications consequent on the use of Lagrange's formula, now, on the other hand, the vogue of the machine seems to be reinstating types of formulae which would once have been dismissed as laborious.

As an example in the use of the formulae we shall interpolate for $\log_{10}\cosh 0{\cdot}365366$, using a short table of eleven-digit entries.

| | | | |
|---|---|---|---|
| 0·360 | 0·0275 | 5462 | 3980 |
| 0·362 | 278 | 5523 | 7805 |
| 0·364 | 281 | 5737 | 9665 |
| 0·366 | 284 | 6104 | 7438 |
| 0·368 | 287 | 6623 | 8989 |
| 0·370 | 290 | 7295 | 2180. |

For Jordan's formula we shall take central data at 0·364, 0·366, with $x = 0{\cdot}683$. For the other formula we shall take the central datum at 0·366, with $x = -0{\cdot}317$. By (19) and (18), we compute

$$g_1 = 283\ 6478\ 4753{\cdot}96, \qquad h_2 = 3\ 0418\ 8143{\cdot}19,$$
$$g_3 = \ldots\ 6630\ 9469{\cdot}22, \qquad h_4 = 6\ 0837\ 4150{\cdot}11.$$
$$g_5 = \ldots\ 6935\ 8892{\cdot}12.$$

The Jordan differences are found to be 152 4715·26 and −7·62. The single difference used in the other formula is −2136·27.

Setting these up on the machine and using values of Gauss coefficients from Chappell's Tables, we find by both formulae the value 0·0283 6461 9695 for the desired interpolate.

The computer, in working by machine, will easily arrange the order of computation in such a way as to avoid the unnecessary clearing from the dials of numbers which enter at the next stage.

A. C. AITKEN.

### III. By G. J. LIDSTONE, LL.D., F.R.S.E.

THE following simple investigation shows the rationale of Jordan's and Aitken's formulae (*ante*, pp. 14 and 18), and proves them to any order of differences with very little algebra.

The underlying principle of interpolation by finite differences to the $t$th order is that the $(t+1)$ given values of $u$ and all the interpolated values lie on the same curve of the $t$th degree.

Since Jordan's and Aitken's formulae depend on symmetry it will be convenient to take the origin at the middle of the range of the given values: in the case of Jordan's formula this is $x=\frac{1}{2}$, and putting $y \equiv x-\frac{1}{2}$, we may write

$$u_x = u_{\frac{1}{2}+y} = F(y) + yf(y),*$$

where $F(y)$ and $f(y)$ are *even* functions of $y$, involving only constants and even powers of $y$. Thus

$$\tfrac{1}{2}(u_{\frac{1}{2}+y} + u_{\frac{1}{2}-y}) = F(y), \quad \text{and} \quad \tfrac{1}{2}(u_{\frac{1}{2}+y} - u_{\frac{1}{2}-y}) = yf(y).$$

If $U_n$ is an approximate value of $u_x$, that is of $u_{\frac{1}{2}+y}$, found by linear interpolation between $u_{\frac{1}{2}-n}$ and $u_{\frac{1}{2}+n}$ for the interpolation interval $(y+n)/2n$, we have

$$\begin{aligned} U_n &= u_{\frac{1}{2}-n} + \frac{y+n}{2n}(u_{\frac{1}{2}+n} - u_{\frac{1}{2}-n}) \\ &= \tfrac{1}{2}(u_{\frac{1}{2}+n} + u_{\frac{1}{2}-n}) + \frac{y}{2n}(u_{\frac{1}{2}+n} - u_{\frac{1}{2}-n})\dagger \\ &= F(n) + yf(n), \end{aligned}$$

*where $y$ is fixed and $n$ varies.* Putting $n=y$, we get

$$U_{n=y} = F(y) + yf(y) = u_{\frac{1}{2}+y} = u_x.$$

Thus $u_x$, that is $U_y$, can be found to any order of differences by interpolation of $U_n$; and since the $F(\ )$ and $f(\ )$ involved will be the same functions that enter into the expression of the $u$'s which are brought into the calculation of the $U$'s, it follows that the interpolated value of $U_y$ will be identical with (and therefore true to the same order of differences as) the value of $u$ which would be got by direct interpolation of the same $u$'s.

Since $U_n$ is an even function of $n$, we have $U_n = U_{-n}$, and $t$ values of $U$ (involving $2t$ values of $u$) are sufficient for interpolation to

* Cf. the Newton-Bessel formula, but for the present purpose it is better to go back to first principles.

† This is the Newton-Bessel formula to the first difference.

$(2t-1)$th differences; also the central differences of $U$ of *odd* order vanish, while the mean central differences of *even* order are equal to the actual differences (of the same order) on each side of the centre. We therefore get the following scheme, which can be easily extended:

$$
\begin{array}{lccccccc}
\vdots & & & & & & & \\
U_{-\frac{7}{2}} = d & & & & & & & \\
 & c-d & & & & & & \\
U_{-\frac{5}{2}} = c & & b-2c+d & & & & & \\
 & b-c & & a-3b+3c-d & & & & \\
U_{-\frac{3}{2}} = b & & a-2b+c & & d-4c+6b-3a & & & \\
 & a-b & & 3b-2a-c & & 5a-9b+5c-d & & \\
U_{-\frac{1}{2}} = a & & b-a & & c-3b+2a & & & \\
 & 0 & & 0 & & 0 & & \\
U_{\frac{1}{2}} = a & & b-a & & c-3b+2a & & & \\
 & b-a & & c-3b+2a & & d-5c+9b-5a & & \\
U_{\frac{3}{2}} = b & & a-2b+c & & d-4c+6b-3a & & & \\
 & c-b & & d-3c+3b-a & & & & \\
U_{\frac{5}{2}} = c & & b-2c+d & & & & & \\
 & d-c & & & & & & \\
U_{\frac{7}{2}} = d & & & & & & & \\
\vdots & & & & & & &
\end{array}
$$

From this it is seen (i) that only one half of the scheme need actually be written down; (ii) Jordan's expressions

$$(b-a),\ (c-3b+2a),\ \dots,$$

are, as pointed out by Dr. Aitken, merely the significant central differences of $U$, of even order, and it would seem that—at least in the absence of a machine—they can be most conveniently calculated as such differences; (iii) the central differences of odd order all vanish.

Interpolating for $U_y$ (with interpolation interval $\frac{1}{2}+y=x$ reckoned from the term preceding the central interval), either by the Newton-Gauss formula or by the Newton-Bessel formula, we get Jordan's formula, (*ante*, p. 16), namely,

$$
\begin{aligned}
U_y = u_x &= U_{\frac{1}{2}} + x_2\Delta^2 U_{-\frac{1}{2}} + (x+1)_4\Delta^4 U_{-\frac{3}{2}} + \dots \\
&= a + x_2(b-a) + (x+1)_4(c-3b+2a) + \dots,
\end{aligned}
$$

where $x_r$ represents the binomial coefficient

$$x(x-1)\dots(x-r+1)/r!.$$

The formula is especially simple and convenient for interpolation to third differences by use of the first two terms.

The above investigation brings out clearly the real nature of Jordan's formula, and involves no troublesome transformations of

coefficients. Aitken's equally important companion formula (*ante*, p. 20), which involves an odd number of $u$'s, can be deduced in a very similar way, but with the origin taken at $x=0$ instead of $x=\frac{1}{2}$, so as to preserve the symmetry which is the essence of the formula and of the proof. For let $V_n$ be the approximate value, based on $u_{-n}$, $u_0$ and $u_n$, and therefore correct to second differences, of $\frac{n}{x}(u_x-u_0)$. Then $V_n=\frac{1}{2}(u_n-u_{-n})+\frac{x}{2n}(u_n-2u_0+u_{-n})$; and if as before the polynomial, which represents the given and interpolated $u$'s, is $F(x)+xf(x)$ (where $F$ and $f$ are even functions of $x$), we get $V_n=nf(n)+\frac{x}{n}[F(n)-u_0]$. This is an odd function of $n$ which changes sign with $n$, vanishes when $n=0$, and takes the value $xf(x)+F(x)-u_0=u_x-u_0$ when $n=x$; thus $u_x=u_0+V_x$. Also, the mean central differences, on a line with $V_0$, will be equal to the adjoining ordinary differences of the same order, and the actual central differences of even order, on a line with $V_0$, will vanish. By writing down a scheme of differences similar to that given for Jordan's formula but involving the values of

$$\dots V_{-2},\ V_{-1},\ V_0=0,\ V_1,\ V_2 \dots,$$

and using either the Newton-Gauss formula or the Newton-Stirling formula, we at once get Aitken's formula, namely,

$$\begin{aligned} u_x &= u_0+V_x=u_0+(V_0=0)+x\Delta V_0+(x+1)_3\Delta^3V_{-1}+\dots \\ &= u_0+xV_1+(x+1)_3(V_2-2V_1)+\dots, \end{aligned}$$

where $V_r$ corresponds to Aitken's $h_{2r}$.

Aitken's formula seems to have the same importance for interpolations involving an odd number of $u$'s, and therefore correct to an even order of differences, as Jordan's has in the case involving an even number of $u$'s. In this connection it may be desirable to point out that for interpolations correct to an even order of differences there is a useful but little-known alternative formula of the Everett type, generally called "Everett II" though it is not due to Everett. *Vide* Steffensen, *Interpolation*, p. 31 [1927; original Danish Edition, 1925]; D. C. Fraser, *Newton and Interpolation*, Newton Memorial Volume, p. 69 [1927]; G. J. Lidstone, *Jour. Inst. Act.* Vol. LX. pp. 351-2 [1929]. In the last-named paper the formula is simply deduced from the Newton-Gauss formulae; and again from Everett's original formula as follows. Everett's original formula may be written (using $x_r$ in the same sense as before)

$$\begin{aligned} v_x &= xv_1+(x+1)_3\Delta^2v_0+(x+2)_5\Delta^4v_{-1}+\dots \\ &\quad -(x-1)v_0-x_3\Delta^2v_{-1}-(x+1)_5\Delta^4v_{-2}+\dots. \end{aligned}$$

If we difference both sides the left-hand side becomes $\Delta v_x$. On the right-hand side the values of $v_0$ and $v_1$ and their central differences are fixed, and the operation of differencing will affect only

the coefficients, which are functions of $x$. Thus, noticing that $\Delta(x+r)_i=(x+r)_{i-1}$, we get at once

$$\Delta v_x=v_1+(x+1)_2\Delta^2v_0+(x+2)_4\Delta^4v_{-1}+\dots$$
$$-v_0-x_2\Delta^2v_{-1}-(x+1)_4\Delta^4v_{-2}-\dots.$$

Putting $\Delta v_x=u_x$ etc., we get "Everett II" in the ordinary form

$$u_x=u_0+(x+1)_2\Delta u_0+(x+2)_4\Delta^3u_{-1}+\dots$$
$$-x_2\Delta u_{-1}-(x+1)_4\Delta^3u_{-2}-\dots,$$

or putting $x=p-\frac{1}{2}$, $1-x=q+\frac{1}{2}$, so that $p=\frac{1}{2}+x$, $q=\frac{1}{2}-x$, we get the form given by Fraser (*loc. cit.*):

$$u_{p-\frac{1}{2}}=u_0+\frac{p^2-\frac{1}{4}}{2!}\Delta u_0+\frac{(p^2-\frac{1}{4})(p^2-\frac{9}{4})}{4!}\Delta^3u_{-1}+\dots$$
$$-\frac{q^2-\frac{1}{4}}{2!}\Delta u_{-1}-\frac{(q^2-\frac{1}{4})(q^2-\frac{9}{4})}{4!}\Delta^3u_{-2}-\dots.$$

G. J. LIDSTONE.

## INTERNATIONAL CONGRESS OF MATHEMATICIANS, 1932.

THE next International Congress of Mathematicians is to be held at Zurich from 4th September to 12th September of this year. The complete programme of lectures and sectional meetings has not yet been received, but the list of those who have already promised to speak is so strong that an attractive week is certain.

Besides the scientific arrangements, there are plans for social functions and excursions; for these Switzerland should prove an ideal setting.

## BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. The names of those sending the questions will not be published.

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**842.** "I never could understand arithmetic", said Mr. Justice Wright during the Kylsant case. And yet arithmetic has certain affinities with the law. It would be a kind of denial of justice to add up two and two to three, or to reckon up one and eightpence as eighteen pence. No doubt there is a new spirit even in mathematics; but anyone trained in the rigours of Euclid cannot get rid of an uneasy sense that Einstein or somebody has been tampering with the scales.—*The Observer*, July 26, 1931.

**843.** An Edinburgh Writer to the Signet has received a legacy of £10,000 to be devoted "to the advancement of astrology". It is rather as if a musician gave a fortune to promote the study of the shawm and the virginals. But, after all, Sir James Jeans's picture of a bleak and four-dimensional universe of mathematical equations is a poor consolation beside the belief that if you are born at the right hour of the right day under the right star you will win the next Dublin sweepstake.—*The Observer*, July 26, 1931.

# THE CORRELATION OF ELEMENTARY TRIGONOMETRY AND GEOMETRY IN ELEMENTARY SCHOOL MATHEMATICS.*

By W. J. Dobbs.

1. The title of my paper is, perhaps, a little misleading. The word *Elementary* is intended to apply to the word *Mathematics*, not to the word *School*, and I will keep strictly to Elementary Mathematics.

The treatment of Geometry in feigned independence of numerical measurement was theoretically abandoned as soon as the supremacy of Euclid collapsed under the onslaught of Professor Perry in the year 1901, and gradually, largely through the efforts of a few outstanding educationists, the teaching of School Geometry in this country has undergone a very remarkable change. We start now with a very much broader foundation, based on intuition and experience, before proceeding to the deductive development which is gradually introduced. The change has come about very slowly, and has been accompanied by some misunderstandings and mistakes. For instance, I notice a tendency in recent years to prolong unduly the preliminary stage and postpone indefinitely the deductive development. This, in my view, is a serious mistake. The broader basis need not cover the whole range of school geometry. Indeed, easy deductions should begin quite early and gradually supersede the procedure adopted in the early stage. Also, in the preliminary course one cannot be too careful to distinguish the properties necessary and sufficient for the construction of a figure from those which arise in consequence. No teacher of geometry can ever afford to lose sight of the fact that the essence of a geometrical training consists in grasping that "If this, then that".

But we are now getting into smoother waters, and the time seems to have arrived for taking a further step towards the unification of mathematical science. I propose in this paper to restrict my remarks to outlining a four-years' course of deductive geometry suitable for schools, terminating at the age, say, of sixteen, and culminating in the first school examination. We will suppose that the pupils have, in a preliminary course, become fully acquainted with the fundamental facts associated with the following topics:

Angles at a point;
Angle conditions for, and angle properties of, parallels;
Angle-sum property of a triangle;
Conditions of congruence of triangles.

2. The deductive development at this stage usually deals with isosceles triangles, parallelograms, proofs of the correctness of the ordinary simple geometrical constructions (perhaps the simple inequalities), and the equal-intercept theorems. Then we have a very suitable opportunity of introducing the cosine of an acute angle. This is by far the most important of the trigonometrical functions, and until it is thoroughly grasped I would mention no other.

* A paper to the London Branch, 28th February, 1931.

I usually introduce the cosine in this way: Let $AB$ and $CD$ be two equal segments of a sloping line, each of length $c$, and let perpendiculars (projectors) be drawn from the points of section upon a horizontal line in the same vertical plane.

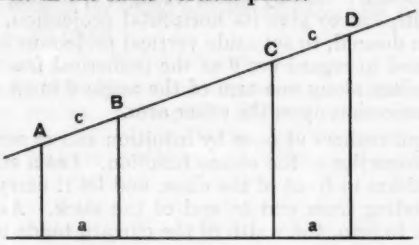


FIG. 1.

It is easily proved that the projections of $AB$ and $CD$ are equal, each, say, of length $a$. It follows that tenths of $c$ along the sloping line project into tenths of $a$, hundredths of $c$ into hundredths of $a$, and so on. Hence it is readily seen that if $AB$ and $PQ$ are any two

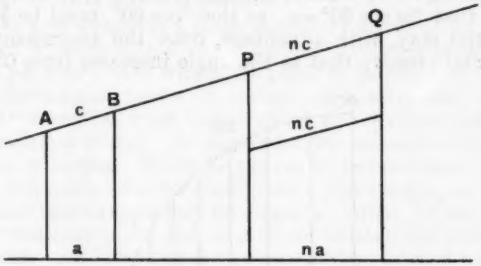


FIG. 2.

segments of the sloping line of length $c$ and $nc$, then their projections are $a$ and $na$ respectively, $n$ being any number expressed in decimals to as many places as we please. Further, since opposite sides of a parallelogram have been proved to be equal, the segment $nc$ may be taken along any line parallel to the original sloping line. Thus

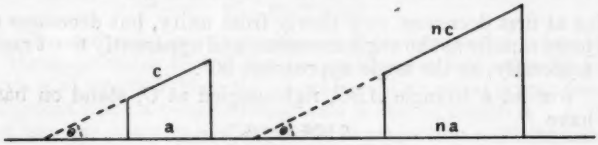


FIG. 3.

it is seen that if one sloping line-segment of length $c$ has a horizontal projection $a$, then another line-segment $nc$, sloping at an equal angle $\theta$, has a horizontal projection $na$, where $n$ is any numerical factor

expressed in decimals to any number of places. Now $a$ is the same fraction of $c$ that $na$ is of $nc$, and this fraction is called the *cosine* of the angle of slope. In this way the pupil learns to regard $\cos\theta$ as the numerical fraction by which a length $c$, sloping at an angle $\theta$, must be multiplied to give its horizontal projection. Later on, it is easy, when desired, to set aside vertical projectors and horizontal projections and to regard $\cos\theta$ as the numerical fraction by which any length taken along one arm of the angle $\theta$ must be multiplied to give its projection upon the other arm.

3. The pupil realizes at once by intuition and experience some of the simple properties of the cosine function. Let a stick be held in a sloping position in front of the class, and let it carry an imagined curtain extending from end to end of the stick. As the angle of slope $\theta$ tends to zero, the width of the curtain tends to

$$1 \times (\text{length-of-stick}) ;$$

as $\theta$ increases from $0°$ to $90°$, the multiplying fraction evidently decreases, and as $\theta$ tends to $90°$ this multiplying fraction tends to zero. Thus the contrariness of the cosine is apparent, and the pupil perceives that $\cos 0° = 1$ and $\cos 90° = 0$.

By standing an equilateral triangle of side $a$ upon one side as base it is seen that $2a \cos 60° = a$, so that $\cos 60°$ must be $\frac{1}{2}$.

The pupil may, with advantage, draw the accompanying figure (Fig. 4), and observe that as the angle increases from $0°$ to $90°$ its

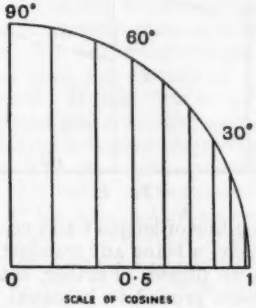


FIG. 4.

cosine at first decreases very slowly from unity, but decreases more and more rapidly as the angle increases, and apparently most rapidly, but uniformly, as the angle approaches $90°$.

4. Now let a triangle $ABC$, right-angled at $C$, stand on base $a$. We have

$$c \cos B = a.$$

Let the triangle be turned on to base $b$, and we have

$$c \cos A = b.$$

Again, let it be turned on to base $c$, and we have

$$a \cos B + b \cos A = c.$$

Multiplying both sides of the last equation by $c$, and making use of the two former equations to eliminate $\cos A$ and $\cos B$, we have

$$a^2 + b^2 = c^2.$$

Again, substituting for $a$ and $b$ in the third equation and dividing both sides by $c$, we have

$$\cos^2 A + \cos^2 B = 1.$$

The last result gives the relation between the cosines of any two complementary angles. Taking 45°, 45° as the two complementary angles, we have $\cos 45° = \frac{1}{2}\sqrt{2}$; taking 30°, 60° as the two complementary angles, we have $\cos 30° = \frac{1}{2}\sqrt{3}$. The pupil can now tabulate as follows :

| Angle - - | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| Cosine - - | 1 | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}$ | 0 |

And it is a good exercise for him to draw the graph of the cosine, from these five known values, for angles ranging between 0° and 90°, and read off the cosines of other angles, comparing them with the values given in a three-figure table.

5. There is now available a most valuable tool which can be used effectively in working numerical examples involving those facts of elementary geometry with which the pupil is dealing. If he is supplied with three-figure tables of cosines, of squares and of square roots, the arithmetical work is but slight and gives added interest to the geometrical course. In particular, he can solve right-angled and isosceles triangles. While he proves in the ordinary way that by making one angle of a parallelogram a right angle, all its other angles become right angles and its diagonals equal, he can now calculate the remaining parts of such a figure constructed according to sufficient data. Similarly, when two consecutive sides of a parallelogram are equal, the parallelogram becomes equilateral, its diagonals bisect the angles at its vertices and meet at right angles ; these give further opportunity for numerical illustration by use of three-figure tables. The pupil learns the use of radial and rectangular Cartesian coordinates and their relations as expressed by the formulae :

$$x = r\cos\theta, \quad y = r\cos(90° - \theta);$$
$$r^2 = x^2 + y^2, \quad \cos\theta = x/r :$$

where all coordinates are at present kept positive and $\theta$ is less than 90°. He appreciates that if the projection of a line-segment upon one axis is of length $a$, then the projection of an equal perpendicular line-segment upon a perpendicular axis is also of length $a$.

And how great a power this notion of projection with its accompanying projective factor gives him ! For instance, if it is desired to construct a square so that its sides may pass successively through four given points $A$, $B$, $C$ and $D$, the problem is to find two perpendicular directions such that the projection of $AC$ upon one may be equal to that of $BD$ upon the other. We therefore draw $AE$

perpendicular and equal to $BD$; then the projections of $AE$ and $AC$ upon the one direction must be equal, and hence the sides of the square must be parallel and perpendicular to $EC$.*

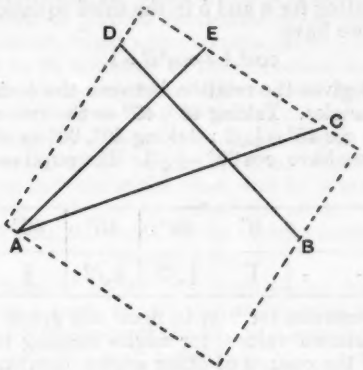


FIG. 5.

6. If desired, all cases of solution of triangles may now be treated, for the machinery is sufficient. But I do not advocate much time being spent on this topic at this stage. In any case, some of the numerical illustrations should be three-dimensional, appeal being made to intuition and experience for the necessary facts of Solid Geometry. A good example is to calculate the size of the earth from measurements of a local map which shows latitude and longitude. Measure a length running due east and west in known latitude, and note the difference of longitude. The cosine of the latitude determines the solution.

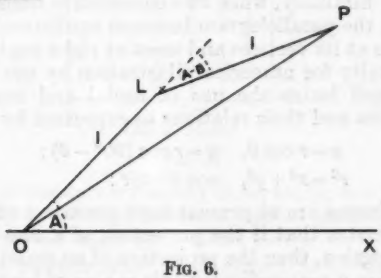


FIG. 6.

It is possible to anticipate other simple properties of the cosine function. For instance, to add two cosines together, one naturally takes $OL$ a unit of length inclined at angle $A$ to $OX$, and then $LP$

* As $AE$ may be drawn in either of two directions, it appears that, in general, the problem admits of two solutions. If the word "successively" be omitted, the number of solutions is, in general, increased to six.

a unit of length inclined at angle $B$ to $OX$ and therefore at angle $A-B$ to $OL$ produced. Then the projection of $OP$ upon $OX$ is $\cos A+\cos B$. It is easily seen that the base angles of the isosceles triangle $LOP$ are each $\frac{1}{2}(A-B)$, so that $OP=2\cos\frac{1}{2}(A-B)$; also that the angle $XOP$ is $\frac{1}{2}(A+B)$. Hence projecting upon $OX$, we have

$$\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).$$

In particular, putting $B=0$, we have

$$1+\cos A=2\cos^2\tfrac{1}{2}A.$$

The last result shows how to calculate $\cos\frac{1}{2}A$ when $\cos A$ is known. Such occasional excursions during a year, say, in which the cosine is the only trigonometrical function considered are most stimulating.

7. In the usual course the next step in the geometry teaching deals with the areas of parallelograms, triangles and rectilinear figures generally. As an illustration of the areas of triangles, consider the following:

With reference to a pair of rectangular axes let the vertices of a right-angled triangle be $A(0, b)$; $B(a, 0)$; $C(0, 0)$. Draw $CD$ perpendicular and equal to $AB$. As the projections of $AB$ upon the

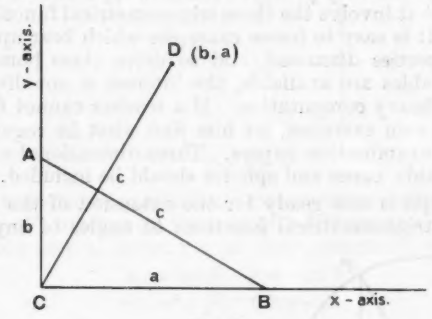


FIG. 7.

axes are $(a, b)$, the projections of $CD$ must be $(b, a)$. Now consider the area $CBDA$. It consists of two triangles $CBD$, $CDA$ of areas $\frac{1}{2}a^2$, $\frac{1}{2}b^2$; but these two triangles have a common base $c$ and are of total height $c$. Hence we deduce

$$\tfrac{1}{2}a^2+\tfrac{1}{2}b^2=\tfrac{1}{2}c^2.$$

The pupil can now, however, master the classical proof of the theorem of Pythagoras. At this stage, also, the meaning and use of the term *locus* may be introduced. In particular, the locus of a moving point which remains equidistant from two fixed points should be discussed, both in its inclusive and exclusive aspects. Similarly, the locus of a moving point which remains equidistant from two fixed lines can be studied.

8. The calculation of the area of a parallelogram or triangle determined by two sides and the contained angle provides a convenient opportunity for introducing the *sine* of an acute angle as an abbreviation for the " cosine of its complement ". It is well, I think, for a while to restrict attention to the cosines and sines of acute angles only. A formula for the area of a parallelogram thus becomes $bc \sin A$ or $bc \sin (180° - A)$ according as the angle $A$ is acute or obtuse, and for a triangle $\frac{1}{2}bc \sin A$ or $\frac{1}{2}bc \sin (180° - A)$. The relation between the cosines of complementary angles, previously discussed, now becomes $\cos^2 A + \sin^2 A = 1$.

When the *sine* has been fully grasped, the *tangent* may be introduced as " sine/cosine " and used in solving right-angled triangles determined by the lengths of the two perpendicular sides. Each of the three functions may now be expressed in terms of each of the other two, but I do not advocate the working of a large number of examples on trigonometrical identities.

9. The simple geometrical properties of the circle usually provide the next topic. Some of these have probably been discussed already as riders involving the properties of isosceles triangles, Pythagoras' Theorem, and the elementary loci. These properties are now treated systematically, and they provide abundant opportunities for numerical exercises that involve the three trigonometrical functions already mentioned. It is easy to frame examples which bear upon the geometrical properties discussed, and so drive them home. And, if three-figure tables are available, the interest is not diverted from the work by heavy computation. If a teacher cannot find time to construct his own exercises, let him find what he requires in past Civil Service examination papers. Three-dimensional examples on planes, pyramids, cones and spheres should be included.

10. The pupil is now ready for the extension of the application of the three trigonometrical functions to angles of any size, and,

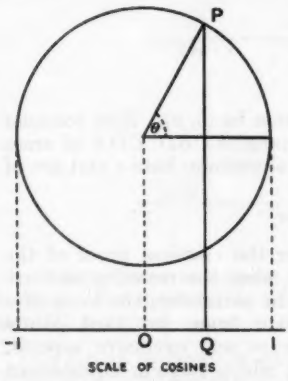


FIG. 8.

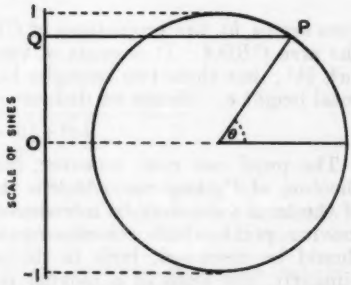


FIG. 9.

particularly, to obtuse angles. What better illustrations can he have of the use of directed numbers? He will easily learn to associate "cos $\theta$" with Fig. 8, "sin $\theta$" with Fig. 9, and "tan $\theta$", meaning "sin $\theta$/cos $\theta$", with Fig. 10.

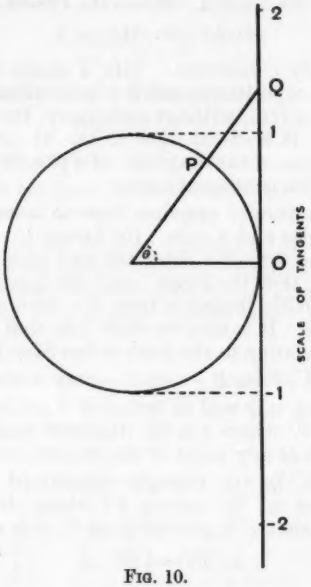


FIG. 10.

The relations between radial and rectangular coordinates, in which the ray $\theta = 0°$ is taken as the axis of $x$ and the ray $\theta = 90°$ as axis of $y$, are now seen to be true in all their generality, namely:

$$x = r\cos\theta, \quad y = r\sin\theta;$$
$$r^2 = x^2 + y^2, \quad \tan\theta = y/x;$$

and Pythagoras' Theorem now takes the form

"Distance-between-points-1-and-2" $= \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$.

11. It has for some years past been the custom in this country to substitute for much of the tedious Second Book of Euclid certain geometrical illustrations of algebraic identities as sufficient for school purposes. I have no objection to this practice, and would extend it still further. Let $ABC$ be any triangle. Take $A$ as origin, $AB$ as axis of $x$, the axis of $y$ at right angles to $AB$, and let the coordinates of the vertices be $A(0, 0)$, $B(c, 0)$, $C(x, y)$.

The equation which expresses the fact that $AC$ is of length $b$ is

$$x^2 + y^2 = b^2.$$

The equation which expresses the fact that $BC$ is of length $a$ is

$$(x - c)^2 + y^2 = a^2.$$

These two equations are true whether $x$ is positive or negative, greater or less than $c$, and, eliminating $y$, we have

$$a^2 = b^2 + c^2 - 2cx.$$

Now in all cases $x = b \cos A$. Hence the *Cosine-Rule for Triangles*:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This rule directly concentrates into a single formula at least a dozen of Euclid's propositions, and it is most illuminating to explore it. It completely solves, without ambiguity, two cases of the solution of triangles. It leads at once to the theorem of Apollonius and to the calculation of the diagonals of a parallelogram determined by two sides and the contained angle.

12. The pupil naturally enquires how to solve a triangle determined by two angles and a side. He knows that when two angles are equal, then the opposite sides are also equal; that when two angles are unequal, then the larger angle has opposite to it the longer side; and he probably imagines that the sides are proportional to the opposite angles. It is easy to show him that this is not so, and to prove the true relation in the form of the *Sine-Rule for Triangles*:

$$a/\sin A = b/\sin B = c/\sin C = \text{circum-diameter}.$$

In this connection it is well to notice that the length of any chord of a circle is $d \sin \theta$, where $d$ is the diameter and $\theta$ the angle which the chord subtends at any point of the circumference on either side.

13. Now let $ABC$ be any triangle determined by $b$, $c$, and $A$, in which $c$ is less than $b$. By taking $AD$ along $AC$ equal to $AB$, as in the usual proof that $B$ is greater than $C$, it is revealed that

$$\angle CBD = \tfrac{1}{2}(B - C).$$

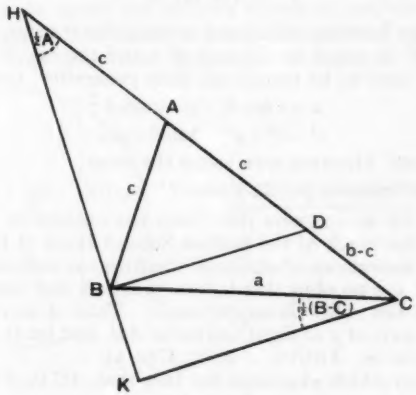


FIG. 11.

Draw $HBK$ perpendicular to $BD$ to meet $CA$ produced at $H$ and the parallel to $DB$ through $C$ at $K$. It is very easy geometry to

prove that

$$HA = AB, \quad \angle H = \tfrac{1}{2}A, \quad \angle HCK = \tfrac{1}{2}(B+C).$$

Hence
$$\begin{cases} KC \tan \tfrac{1}{2}(B-C) = BK = (b-c)\cos \tfrac{1}{2}A, \\ KC \tan \tfrac{1}{2}(B+C) = HK = (b+c)\cos \tfrac{1}{2}A. \end{cases}$$

Thus
$$\frac{\tan \tfrac{1}{2}(B-C)}{\tan \tfrac{1}{2}(B+C)} = \frac{b-c}{b+c}.$$

This is the *Tangent-Rule for Triangles.*

Since
$$\tfrac{1}{2}(B+C) = 90° - \tfrac{1}{2}A, \quad \text{.....(i)}$$

a table of tangents determines $\tfrac{1}{2}(B-C)$. .....(ii)

From (i) and (ii) we have both $B$ and $C$ by a more convenient method than the use of the cosine-rule.

14. A short course on Similar Triangles, in which elementary trigonometry is freely used, will carry the pupil up to the first school examination and prepare the specialist for an advanced course. The *Proportion Property of Parallels* should now, I think, take the following form :

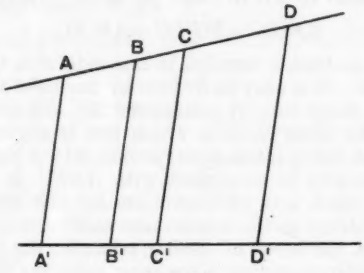


FIG. 12.

Let $AB$ and $CD$ be any two segments of a straight line, and let any parallel lines $AA'$, $BB'$, $CC'$, $DD'$ be drawn to meet any other transversal at $A'$, $B'$, $C'$, $D'$. Let $CD$ be divided into any integral number $n$ of equal parts each of length $h$, and let parts each equal to $h$ be set off successively along $AB$. Let parallels to $AA'$ be drawn through the points of section to meet $A'D'$. These parallels will divide $C'D'$ into $n$ equal parts, each $h'$, and will cut off successively equal parts, each $h'$, along $A'B'$. In the particular case when it happens that $AB$ consists of an integral number $m$ of equal parts $h$, $A'B'$ consists of the same number $m$ of equal parts $h'$, and in this case $AB/CD = A'B'/C'D'$. But in the general case

$$mh < AB < (m+1)h$$

and then
$$mh' < A'B' < (m+1)h'.$$

In this case
$$m/n < AB/CD < (m+1)/n$$

and
$$m/n < A'B'/C'D' < (m+1)/n.$$

Now the integer $n$ is as large a number as we please to make it, so that the fractions $m/n$ and $(m+1)/n$ differ by $1/n$, which is as small a fraction as we please. We infer that

$$AB/CD = A'B'/C'D'.$$

15. I prefer to preface the discussion of the conditions for the similarity of two triangles by the following proposition.

Let $ABC$ be any triangle, and let a parallel to $BC$ cut $AB$ at $E$ and $AC$ at $F$. Then $AE/AB = AF/AC$.

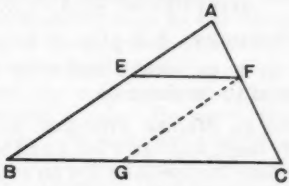


FIG. 13.

Draw $FG$ parallel to $AB$ to meet $BC$ at $G$. Then

$$EF/BC = BG/BC = AF/AC.$$

Hence it is seen that a parallel to one side of a triangle cuts off a similar triangle. It is easy to deduce by means of this preliminary proposition the three sets of conditions for the similarity of two triangles from the three cases in which two triangles are known to be congruent, $AE$ being made equal to one side of the other triangle which is then seen to be congruent with $AEF$. It should here be noted that the three sets of conditions for the similarity of two triangles are apparent in the cosine- and sine-rules for triangles.

The proof that the areas of similar triangles are proportional to the squares on corresponding sides may take the following form: let $A, B, C$; $a, b, c$ be the elements of one triangle. Then $A, B, C$; $na, nb, nc$ may be taken as the elements of the other. Hence

$$\Delta_1/\Delta_2 = \tfrac{1}{2}bc \,.\, \sin A / \tfrac{1}{2}nb \,.\, nc \,.\, \sin A = 1/n^2.$$

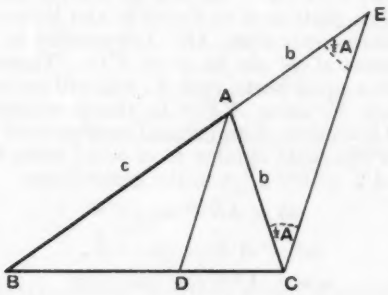


FIG. 14.

After proving, in the usual way, that the bisector of the vertical angle of a triangle divides the base in the proportion of the sides, we may very well write down expressions for the lengths of the lines in the figure, (Fig. 14). Thus :

$$BD/c = DC/b = a/(b+c);$$
$$AD/c = CE/(b+c) = 2b \cdot \cos \tfrac{1}{2}A/(b+c).$$

Hence, also

$$\begin{aligned} AB \cdot AC - BD \cdot DC &= bc\,[1 - \{a/(b+c)\}^2] \\ &= bc \cdot (2bc + b^2 + c^2 - a^2)/(b+c)^2 \\ &= bc \cdot 2bc\,(1 + \cos A)/(b+c)^2 \\ &= 4\left(\frac{bc}{b+c}\right)^2 \cos^2 \tfrac{1}{2}A \\ &= AD^2. \end{aligned}$$

*Ptolemy's Theorem* should, I think, be proved in a way which also establishes its converse. Make

$$\angle BAE = \angle CAD \quad \text{and} \quad \angle ABE = \angle ACD.$$

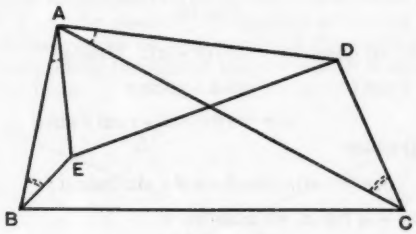


FIG. 15.

Then, in general, $BE + ED$ is greater than $BD$, but is equal to $BD$ when, and only when, $ABCD$ is cyclic. From the similarity of the triangles $ABE$ and $ACD$ we have

$$AB \cdot CD = BE \cdot AC. \quad \text{.....(i)}$$

and also $AB/AE = AC/AD$. Now $AB$, $AC$ and $AE$, $AD$ contain equal angles. Hence $ABC$ and $AED$ are also similar triangles, and thus

$$BC \cdot AD = ED \cdot AC. \quad \text{.....(ii)}$$

Whence from (i) and (ii) we have

$$AB \cdot CD + BC \cdot AD = (BE + ED) \cdot AC.$$

Thus $AB \cdot CD + BC \cdot AD$ is, in general, greater than $BD \cdot AC$ but becomes equal to $BD \cdot AC$ when, and only when, $ABCD$ is cyclic. The trigonometrical interpretation of Ptolemy's Theorem is well known, and I need not refer to it.

16. In conclusion let us consider the geometrical meaning of the expression $x\cos\alpha+y\sin\alpha$. Using radial coordinates, let $OX$, $OY$, $OV$ be the rays $\theta=0°$, $\theta=90°$, $\theta=\alpha$, respectively. Let $P$ be the point whose coordinates are $(x, y)$ with reference to the Cartesian axes $OX$, $OY$. Then the expression $x\cos\alpha+y\sin\alpha$ gives in magnitude and sense in all cases the projection of $OP$ upon $OV$. Now let $(r, \theta)$ be the radial coordinates of $P$. Then the same projection

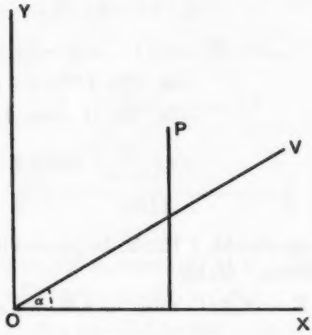


FIG. 16.

is also given in all cases by $r\cos(\theta-\alpha)$. Hence

$$r\cos(\theta-\alpha)=x\cos\alpha+y\sin\alpha$$
$$=r\cos\theta\cos\alpha+r\sin\theta\sin\alpha.$$

Hence in all cases

$$\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha; \quad \text{.....(i)}$$

Substituting $-\alpha$ for $\alpha$, we also have

$$\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha. \quad \text{.....(ii)}$$

In each of (i) and (ii), substituting $90°-\theta$ for $\theta$, we have

$$\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha, \quad \text{.....(iii)}$$
$$\sin(\theta-\alpha)=\sin\theta\cos\alpha-\cos\theta\sin\alpha. \quad \text{.....(iv)}$$

The proof is quite general and establishes these four fundamental formulae for all cases.

I venture to think that some such course as I have sketched would lead more rapidly to the acquisition of knowledge, without presenting more difficulty than the course at present generally adopted.

W. J. DOBBS.

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**844.** For when she came to think it out alone she realised that no spiritual geometry could square the triangle of incompatibility or reconcile the divergent lines of force that composed it.—Francis Brett Young, *Portrait of Clare.* [Per Mr. E. Lax.]

## MATHEMATICAL NOTES.

**1016.** *Note on the difficulty of a geometric construction* (XV, p. 393).

I do not commit myself as to which of the two operations—to draw a circle centre $C$ through $A$ ; or to draw the line $CA$—the student would think the easier, for the simple reason that I do not think it possible to give a definite answer to the question.

On the one hand, it is easier to slide a set-square over a drawing-board for the purpose of joining two clearly defined points than it is to draw an arc—of about $\frac{1}{4}$ inch radius—with given centre to pass through a given point.

On the other hand, it is more difficult to join the point of intersection of two nearly coincident lines to the point of intersection of two nearly coincident lines than to draw an arc—of about 4 inches radius—with given centre and to pass through a given point.

B. NAYLOR.

**1017.** *To find the criteria determining* (*i*) *whether a point is within one of the obtuse or within one of the acute angles between two lines and* (*ii*) *whether two points are within the same or opposite obtuse angles or within the same or opposite acute angles between two lines.*

If $PM$ be drawn perpendicular to one line to meet the other in $A$, then $PM \,.\, PA$ will be positive or negative according as $P$ is within one of the obtuse angles or within one of the acute angles between the lines. Completing the figure, the angle at $P$ is equal to the angle $\alpha$ between the lines.

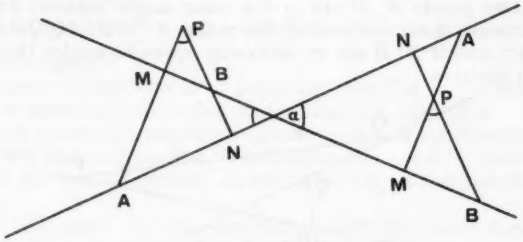



FIG. 1.

1. If the coordinates of the point $P$ are $(x_1, y_1)$ and the equations of the lines are $L \equiv ax+by+c=0$ ; $L' \equiv a'x+b'y+c'=0$,

$$PM \,.\, PA = PM \,.\, PN \sec \alpha$$
$$= \frac{L_1}{\sqrt{(a^2+b^2)}} \cdot \frac{L_1'}{\sqrt{(a'^2+b'^2)}} \cdot \frac{\sqrt{(a^2+b^2)} \,.\, \sqrt{(a'^2+b'^2)}}{aa'+bb'}$$
$$= L_1 L_1' / (aa'+bb').$$

and the sign of this is the required criterion.

*Cor.* The bisectors of the angles between $L=0$ and $L'=0$ are

$$L/\sqrt{(a^2+b^2)} = \pm L'/\sqrt{(a'^2+b'^2)},$$

where the square roots are taken each with the positive sign; if $(x_1, y_1)$ is a point on one of these, the criterion for the internal or external bisector is the sign of

$$\pm L_1'^2 \sqrt{\left(\frac{a^2+b^2}{a'^2+b'^2}\right)} \Big/ (aa'+bb') :$$

that is, if $aa'+bb'$ is positive, the plus sign gives the external bisector, and so on.

2. If the equation of the lines is given in the form

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

the equation of the lines $PM$, $PN$ is

$$T \equiv b\,(x-x_1)^2-2h\,(x-x_1)\,(y-y_1)+a\,(y-y_1)^2=0.$$

The equation to the circle $AMBN$ is

$$S+T \equiv (a+b)\,(x^2+y^2)+\ldots=0.$$

Hence
$$PM\,.\,PA=(S_1+T_1)/(a+b)$$
$$=S_1/(a+b),$$

since
$$T_1 \equiv 0.$$

*Cor.* The criterion that the hyperbola $S=0$ should lie within the obtuse or within the acute angle between its asymptotes $S-\Delta/C=0$ is

$$(S_1-\Delta/C)/(a+b) \gtrless 0$$

where $(x_1, y_1)$ is a point on the hyperbola. But $S_1=0$ and $C$ is negative; hence the criterion is $\Delta/(a+b) \gtrless 0$.

3. If two points $A$, $B$ are in the same angle between two lines, and $AB$ meets them in $P$ and $Q$, the ratios $AP/PB$, $AQ/QB$ are both negative; and if $A$, $B$ are in vertically opposite angles these ratios are both positive.

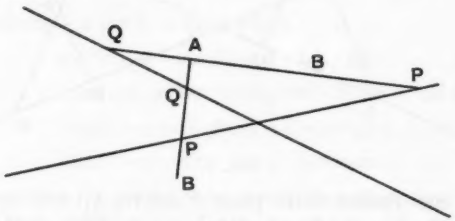



FIG. 2.

Calling either ratio $m : l$, and the points $A$, $B$ $(x_1, y_1)$ and $(x_2, y_2)$, the point

$$\{(lx_1+mx_2)/(l+m),\ (ly_1+my_2)/(l+m)\}$$

is on the lines $S=0$, so that as usual

$$l^2S_1+2lmP_{12}+m^2S_2=0,$$

or
$$l^2\,.\,\frac{S_1}{a+b}+2lm\,.\,\frac{P_{12}}{a+b}+m^2\,.\,\frac{S_2}{a+b}=0.$$

Hence the conditions that $(x_1, y_1)$ and $(x_2, y_2)$ may be within the same obtuse angle between the lines $S=0$ are that these three coefficients should be all positive, and that they may be within the same acute angle, all three must be negative. For opposite obtuse angles, $S_1/(a+b)$ and $S_2/(a+b)$ must be positive and $P_{12}/(a+b)$ negative, while for opposite acute angles, $S_1/(a+b)$, $S_2/(a+b)$ must be negative, $P_{12}/(a+b)$ positive.

*Cor.* Taking as the given lines $S-\Delta/C=0$, that is, the asymptotes of the hyperbola $S=0$, the criterion determining whether $(x_1, y_1)$ and $(x_2, y_2)$ are on the same or opposite branches thereof is that the roots of the equation

$$l^2(S_1-\Delta/C)+2lm(P_{12}-\Delta/C)+m^2(S_2-\Delta/C)=0$$

are both negative or both positive. Since $S_1=S_2=0$, we obtain

$$l^2+2lm(1-CP_{12}/\Delta)+m^2=0,$$

and the required criterion is $CP_{12}\leqq\Delta$. N. M. GIBBINS.

1018. *Tannery's Theorem.*

Prof. Neville has observed (Note 963, *Gazette*, July 1930) that the difficulty of this theorem is commonly exaggerated, and suggests a simpler version of the theorem. It is not, however, at first sight obvious that the usual condition of a dominating series secures his condition, that $\sum_{s=1}^{r} v_s(n) \to \sum_{s=1}^{r} w_s$ as $n \to \infty$, uniformly with respect to $r$. Further, the applications usually require that the existence of $\lim_{n\to\infty} f(n, n)$ should be deduced from that of $\lim_{r\to\infty} g(r)$, *i.e.* $\sum^{\infty} w_s$, and not *vice versa*.

The object of this note is to point out that Tannery's theorem is really the same as that which states that the sum of a uniformly convergent series of continuous functions of $x$ is itself continuous; in fact, the theorem asserts "continuity for $n=\infty$", and follows from the theorem of continuity for finite $x$ by a natural change of variable.

I give the proof by this method (under the usual condition), following the notation of Bromwich (*Infinite Series*, 2nd edition, page 136):

*Let* $F(n)=\sum_{r=0}^{P(n)} v_r(n)$ *where*

(i) $P$ *is an increasing function of* $n$, *which* $\to\infty$ *as* $n\to\infty$,

(ii) *for any given* $r$, $\lim_{n\to\infty} v_r(n)=w_r$,

(iii) $|v_r(n)|<M_r$ *for all* $n$, *where* $\Sigma M_r$ *converges.*

*Then* $F_i(n)\to\sum^{\infty} w_r$ *as* $n\to\infty$.

The convergence of $\Sigma w_r$ follows from (ii) and (iii), which give

$$|w_r| \leq M_r.$$

(*a*) Define $f_r(n) = v_r(n)$ if $r \leq P(n)$; $f_r(n) = 0$ if $r > P(n)$.

$$f_r(x) = (x-n)f_r(n+1) + (n+1-x)f_r(n) \text{ for } n \leq x \leq n+1.$$

Clearly $f_r(x)$ is a continuous function of $x$, which $\to w_r$ as $x \to \infty$, and $|f_r(x)| \leq M_r$.

Also for $x$ integral, $x = n$,

$$\sum_{r=0}^{\infty} f_r(x) = \sum_{r=0}^{\infty} f_r(n) = \sum_{r=0}^{P} v_r(n) = F(n).$$

(*b*) Write $\quad \phi_r(y) = f_r\left(\frac{1}{x}\right), \quad 0 < y \leq 1.$

$$\phi_r(0) = w_r.$$

Then $\phi_r(y)$ is certainly continuous for $0 < y \leq 1$; also, since

$$\lim_{y \to +0} \phi_r(y) = \lim_{x \to \infty} f_r(x) = w_r = \phi_r(0),$$

$\phi_r(y)$ is continuous in $0 \leq y \leq 1$, and $|\phi_r(y)| \leq M_r$;

hence $\sum_{r=0}^{\infty} \phi_r(y)$ converges uniformly in $0 \leq y \leq 1$.

By the theorem referred to above,

$$\lim_{y \to 0} \sum_{0}^{\infty} \phi_r(y) = \sum_{0}^{\infty} \phi_r(0) = \sum_{0}^{\infty} w_r,$$

$$i.e. \lim_{x \to \infty} \sum_{0}^{\infty} f_r(x) = \sum_{0}^{\infty} w_r.$$

Choosing the integral values of $x$ for which $\sum_{0}^{\infty} f_r(n) = F(n)$,

$$\lim_{n \to \infty} F(n) = \sum_{0}^{\infty} w_r.$$

A. J. WARD.

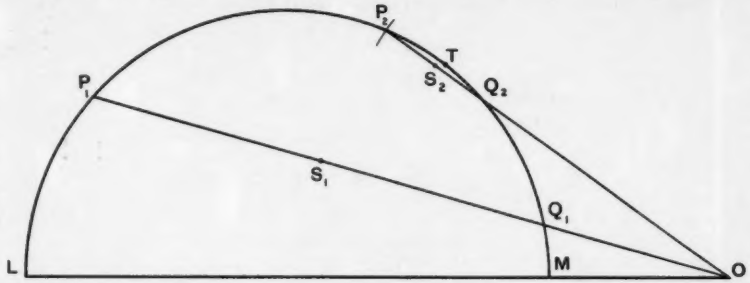
1019. *An iteration in terms of Euclidean Geometry—Newton's Rule.* In ch. vi. of their *Calculus of Observations*, Whittaker and Robinson give the following rule for obtaining the square root of a number, $N$. Take any number $x_0$ and form the sequence

$$x_0, \; x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right), \; \ldots \; x_{t+1} = \frac{1}{2}\left(x_t + \frac{N}{x_t}\right).$$

Then the sequence $\{x_t\}$ tends to a limit which is $\sqrt{N}$.

The geometry of this rule for finding the numerical value of the square root of a positive number $N$ does not appear to have been observed. It has the advantage of making the nature of the operation intelligible to schoolboys.

Describe a circle of radius $\frac{1}{2}(N-1)$, and produce the diameter $LM$ to $O$, so that $MO=1$. Take any length $x_0$ and draw a secant




$OP_1$ of length $x_0$, cutting the circle again at $Q_1$. Bisect $P_1Q_1$ at $S_1$. Then we have

$$\begin{aligned} OP_1 \,.\, OQ_1 &= OL \,.\, OM = N, \\ x_0 \,.\, OQ_1 &= N, \\ OS_1 &= \tfrac{1}{2}(OP_1 + OQ_1) \\ &= \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) \\ &= x_1. \end{aligned}$$

Now take $OP_2 = OS_1 = x_1$, and repeat the construction with $OP_2$ in place of $OP_1$, and a succession of points $P_1, P_2, P_3, \ldots$ approaching $T$ is obtained, with a second sequence $Q_1, Q_2, Q_3, \ldots$ representing $N/x_0, N/x_1, N/x_2 \ldots$ also tending to $T$. If $N$ is less than 1, then $L$ lies between $O$ and $M$ and a circle of radius $\frac{1}{2}(1-N)$ is drawn.

The following connection between the arithmetic theory of limits and the idea of the tangent as a limit may be noted.

For any two positive numbers, the arithmetic mean is greater than the geometric. Hence

$$\frac{1}{2}\left(x_t + \frac{N}{x_t}\right) > \sqrt{x_t \,.\, \frac{N}{x_t}},$$

or

$$x_{t+1} > \sqrt{N}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Moreover the arithmetic mean of two positive numbers lies between them. Thus

$$x_t > \frac{1}{2}\left(x_t + \frac{N}{x_t}\right) > \frac{N}{x_t}$$

or

$$x_t > x_{t+1},$$

provided $x_t$ is greater than $\sqrt{N}$, which is clearly the case for all $x_t$ by (i) save possibly $x_0$. Hence $\{x_t\}$ is a decreasing sequence (with the possible but trivial exception of its first term); $\{N/x_t\}$ is an increasing sequence: the two sequences have the common limit $\sqrt{N}$.

H. Peat.

1020. *Perigal's Dissection for the Theorem of Pythagoras.*

The idea for the following note was suggested to me by a figure sent to me by Miss M. Charlesworth, aged sixteen.

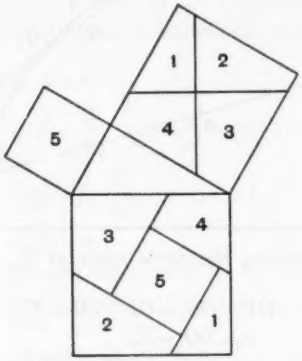



FIG. 1.

In Perigal's dissection for the theorem of Pythagoras the parallel and perpendicular to the hypotenuse are drawn through the centre of the larger of the two squares on the sides ; but it is not necessary

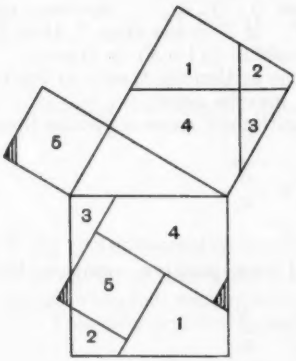



FIG. 2.

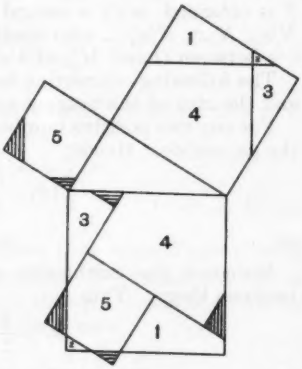



FIG. 3.

for the lines to be drawn through that point, as the figures show. Other interesting cases arise when the square on the smaller side is divided up ; again there are several cases. Anyone can draw the figures for himself.

A. W. SIDDONS.

**1021.** *A Proof of the Fundamental Theorem of Farey Series.*

The following proof * of the fundamental theorem of Farey series seems to the writer simpler than any hitherto published. We recall that the Farey series of order $m$, where $m$ is a positive integer, is the set of all irreducible fractions (with positive denominators), none of whose denominators exceeds $m$, arranged in order of magnitude. Thus, the Farey series of order 3 is

$$\cdots, -\frac{1}{1}, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \cdots.$$

The theorem in question is :

*If* $\frac{p}{q}$ *and* $\frac{P}{Q}$ *are two consecutive terms in the Farey series of order* $m$, $\left(\frac{p}{q} < \frac{P}{Q}\right)$, *then* $qP - pQ = 1$.

This we prove as follows. Let $Q$ be the root of the congruence $1 + px \equiv 0$, mod $q$, which satisfies the condition (1) $m - q < Q \leq m$, and let $P = \frac{1+pQ}{q}$, whence (2) $qP - pQ = 1$. Then $\frac{P}{Q}$ is a term of the Farey series, for $0 < Q \leq m$ and $(P, Q) = 1$ in view of (2). $\frac{P}{Q} > \frac{p}{q}$ also follows from (2).

Finally, let $\frac{p'}{q'}$ be any fraction at its lowest terms, such that $\frac{p}{q} < \frac{p'}{q'} < \frac{P}{Q}$. Then

$$q' = q'(qP - pQ) = Q(p'q - pq') + q(Pq' - p'Q) \geq Q + q > m.$$

Hence, $\frac{p'}{q'}$ is not a term in the Farey series and $\frac{p}{q}$ and $\frac{P}{Q}$ are consecutive terms ; this fact, in connection with relation (2), yields the theorem.

W. L. G. Williams.

McGill University, Montreal, Canada, August 10, 1931.

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**845.** He [Charles James Fox] gave himself diligently to mathematics, which he liked "vastly". "I believe they are useful," he writes, "and I am sure they are entertaining, which is alone enough to recommend them to me."—G. O. Trevelyan, *The Early History of Charles James Fox*, p. 55 (1881).

**846.** It was as impossible for me, as for any other person, to hold communication with [De Morgan] on Mathematical questions, and avoid deriving great advantage from his sagacity and erudition in Mathematics. I have not, I trust, abused these advantages by appropriating to myself anything which belongs to him ; but I have endeavoured, while possessing those advantages, to carry on my researches with originality and independence.—Henry Warburton, in the Introduction to a paper, *Trans. Cambridge Phil. Soc.* 8, 1847.

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* The present proof is a simplification of my earlier proof (*Trans. of the Royal Society of Canada*, Third Series, vol. xxi. 1927, pp. 85-87). Landau uses the same method in his *Vorlesungen* (1927), vol. i. pp. 98-99.

## REVIEWS.

**Geometry of Four Dimensions.** By A. R. FORSYTH. Two volumes. Pp. xxx + 468 and xii + 520. 75s. net. 1930. (Camb. Univ. Press.)

This monumental work treats in a thorough and masterly manner the geometry of Euclidean space of four dimensions and its various sub-spaces. The method employed in connection with curved configurations is substantially an amplification of that devised by Gauss in his treatment of curved surfaces in three-space. This does not imply, however, that the results obtained are merely a generalisation of the ideas and formulae of classical Differential Geometry. The additional dimension makes the subject a much broader one. The author introduces many new geometrical concepts, and discusses them with the thoroughness and depth that we should expect of him. The thousand pages contained in the two volumes represent an enormous amount of clear thinking and skilful research, the net result of which is a substantial addition to mathematical knowledge and a valuable contribution to mathematical literature.

The plenary Euclidean space $S_4$, of four dimensions, is referred to by the author as quadruple homaloidal space, the term homaloidal having the significance of flat or uncurved. Manifolds or varieties are spoken of as *amplitudes*. A sub-space of three dimensions the author calls a *region*, and an uncurved region a *flat*. These are commonly known as a hypersurface of $S_4$ and a hyperplane respectively. An amplitude of two dimensions is called a *surface* in agreement with the usual practice, and an uncurved surface a *plane*. An amplitude of one dimension is, of course, a *curve*, which is a *straight line* in the particular case when its curvature is zero. The point coordinates employed are Cartesian coordinates $x, y, z, v$, referred to four orthogonal axes $OX$, $OY$, $OZ$, $OV$ and six coordinate planes $YOZ$, $ZOX$, $XOY$, $XOV$, $YOV$, $ZOV$. Uncurved configurations are represented by equations which are linear in the coordinates.

The whole book may be divided into five chief sections.

The *first* of these sections, which comprises the first seven chapters and is largely preparatory in scope, deals with uncurved sub-spaces of $S_4$—straight lines, planes and flats. A straight line is represented by three linear equations, a plane by two and a flat by one. Thus a straight line may be regarded as the intersection of three flats, or as the intersection of a flat and a plane. A number of the results found for straight lines and flats are natural extensions of the known properties of straight lines and planes in Euclidean three-space. But many interesting results are proved, especially in connection with planes. Thus an arbitrary straight line does not meet an arbitrary plane. Perpendiculars from two external points to a plane are parallel only when these points lie on a straight line which meets the plane. Two arbitrary planes in $S_4$ meet in a point; but planes in the same flat meet in general in a straight line. There is a unique normal direction to a flat, and parallel flats have parallel normals. The inclination of two flats is the inclination of their normals. It is shown, on the other hand, that, at a point $P$ of a given plane, there is no unique direction in $S_4$ perpendicular to the plane. The locus of straight lines at $P$ perpendicular to the given plane is another plane, which is said to be *completely orthogonal* to first plane, every direction in either plane being perpendicular to every direction in the other. Two planes, which are not completely orthogonal, are said to be *perpendicular* when it is possible to select a unique direction in either plane perpendicular to every direction in the other. It is proved that this relation is a reciprocal one. Further, the inclination of two given planes is defined in connection with the projection upon either plane of a closed figure in the other, the cosine of the inclination being the ratio of the area of the projection to that of the original figure. The orientation of a plane is specified by six *orientation coordinates*, which are the cosines of its inclinations to the six coordinate planes.

The final chapter of this first section deals with globular representation and rotations in $S_4$. Prof. Forsyth shows that, for rotations in quadruple space, the fundamental element is rotation round a plane, producing a displacement of perpendicular axes in the orthogonal plane. Rotation round a point is composite, since the most general rotation of that kind, leaving an orthogonal frame undeformed, can be compounded of rotations round planes. An orthogonal frame of four axes, the origin being fixed, can be changed from any position to any other position by two suitable rotations round two orthogonal planes, the rotation round either plane being independent of that round the other.

The *second section* is devoted mainly to the intrinsic geometry of skew curves in $S_4$. The coordinates $x, y, z, v$ of the current point $P$ are known functions of the arc-length $s$ of the curve. The system is referred to a quadruple orthogonal frame $PT, PC, PB, PF$ consisting of the tangent, principal normal, binormal and trinormal of the curve at $P$. The tangent, principal normal and osculating plane, the circle of curvature and the centre $C$ and radius of curvature are defined substantially as in the case of three dimensions. The *normal flat* of the curve at $P$ is the flat through $P$ normal to the tangent. The principal normal $PC$ is the intersection of this flat by the osculating plane. The *osculating flat* at $P$ is the flat determined by four consecutive points of the curve. The binormal $PB$ forms with the tangent and the principal normal a right-handed system of orthogonal directions in this flat. The *trinormal* $PF$ is normal to the osculating flat at $P$. The plane of $PC$ and $PB$ is called the *normal plane* at $P$, and that of $PB$ and $PF$ the *orthogonal plane*. The latter is completely orthogonal to the osculating plane.

As in the case of a curve in triple space, the torsion is defined as the arc-rate of turning of the osculating plane. Similarly the *curvature of tilt*, or briefly, the *tilt*, is the magnitude of the arc-rate of turning of the trinormal. If the tilt is everywhere zero the curve is contained in a flat; for then the osculating flat is one and the same all along the curve. The *sphere of curvature* at $P$ is defined as the sphere, lying in the osculating flat, which passes through four consecutive points of the curve at $P$. Its centre $S$ and its radius are called respectively the *centre* and *radius of spherical* (or *second*) *curvature*. The globe which passes through five consecutive points at $P$ is called the *globe of curvature*. Its centre $G$ and its radius are the *centre* and *radius of globular* (or *third*) curvature. The sphere of curvature is the intersection of the globe of curvature by the osculating flat; and the circle of curvature is the intersection of either by the osculating plane. In terms of these fundamental concepts Prof. Forsyth develops the geometry of curves in considerable detail, proving many interesting and important results. The loci of the centres of circular, spherical and globular curvature are examined at length, and the extension of the Serret-Frenet formulae to the present case is effected.

This section contains also an excellent piece of work on the envelopes of the flats and planes associated with the curve at any point. It is shown that the envelope of the osculating flat is a developable region, *i.e.* one that can be deformed into a flat without extension or rupture. This is called the *osculating developable region*. It contains the osculating developable surface, which is the envelope of the osculating planes. Similarly the envelope of the normal flat is the *normal developable region*, containing the envelope of the normal plane, whose edge of regression is the locus of the globular centre $G$, called the *globular centric*. The author proves that the tangent, principal normal, binormal and trinormal of the globular centric are parallel respectively to the trinormal, binormal, principal normal and tangent to the given curve; and expressions are found for the curvature, the torsion and the tilt of the globular centric. Finally, to this section is added a chapter outlining the complete aggregate of results, analytical and descriptive, for curves in Euclidean space of $n$ dimensions.

The *third section* is concerned with surfaces in the quadruple space. Prof. Forsyth stresses the fact that the geodesics of the surface are the fundamental elements, and he investigates their properties in considerable detail. The differential equations satisfied by geodesics are found, and formulae are proved for the curvature, the torsion and the tilt of a geodesic. At a point $P$ of the surface there is a unique tangent plane, and also a unique *orthogonal plane*, which is the plane through $P$ completely orthogonal to the tangent plane. These two planes are fundamental planes of reference. There is no unique normal direction to the surface at a point; but the principal normal of the geodesic through $P$ in a given direction is called the *normal* to the surface corresponding to that direction, and the curvature of the geodesic is the *normal curvature* of the surface in this direction. For different directions through $P$ the normals to the surface are different; but they all lie in the orthogonal plane at $P$. It is shown that there are four families of lines of curvature on the surface. The directions of the lines of curvature at a point, called the principal directions of curvature, are those corresponding to the maximum and minimum values of the normal curvature at that point. They are also the directions which give zero torsion of the geodesic tangent. When the surface is contained in a flat the number of lines of curvature through a point is only two, for the problem is then that of a surface in Euclidean three-space.

The author also investigates at some length the properties of two loci associated with the surface at a point $P$. The first of these is the locus of the centre $G$ of circular curvature of a geodesic through $P$. It is shown that this locus lies in the orthogonal plane, and is a curve of the lemniscate type. The four positions of $G$ corresponding to the principal directions at $P$ are the feet of the four normals from $P$ to the lemniscate. The second locus considered is that of the *orthogonal centre* for different directions in the surface at $P$, the orthogonal centre $\Gamma$ for a given direction being the point of intersection of the orthogonal planes at $P$ and at a consecutive point of the surface in this direction. The length $P\Gamma$ is called the *orthogonal radius* for this direction. The locus of $\Gamma$ for different directions at $P$ is a conic in the orthogonal plane, called by Kommerell the *characteristic* of the surface at $P$. Prof. Forsyth makes more than one reference to Kommerell's work in this field. The properties of this conic are discussed, and it is shown that the conic and the lemniscate touch each other at the four points corresponding to the principal directions at $P$, and also that the lemniscate is the pedal of the conic with respect to $P$.

The *fourth section*, which occupies the greater part of Volume II, is concerned with regions, *i.e.* curved spaces of three dimensions within the enveloping $S_4$. The author emphasises the fundamental importance of the geodesics of the region, whose properties are studied in detail. Further, it is shown that there is a unique normal at each point of the region. Any direction in the region at a point $P$ determines a plane containing this direction and the normal at $P$; and the circular curvature of the curve of intersection of the region and this plane is called the *linear curvature* of the region at $P$ for the direction considered. It is identical with the circular curvature of the regional geodesic through $P$ in that direction. There are three principal directions at each point corresponding to the maximum and minimum values of the linear curvature of the region at that point, and these are mutually perpendicular. The lines of curvature are the three congruences of curves whose directions at any point are these principal directions.

The properties of geodesic surfaces of the region are also considered, and expressions are found for their Gaussian and "mean" curvatures at a point $P$. The former is, of course, the Riemannian curvature of the region for the orientation of the surface at $P$, while the latter appears to have been neglected by earlier writers. The author shows that there are three principal orientations in the region which give maximum and minimum values of these superficial curvatures, and that these orientations are identical with those determined by

the principal directions of linear curvature. Also the principal values of the superficial curvatures at $P$ are the product and the sum of the linear curvatures of the region for the two principal directions which lie in the tangent plane to the geodesic surface at $P$. The measure of volumetric curvature of the region at any point is defined as the product of the three principal linear curvatures. It is not associated with any direction or orientation. In a separate chapter these results are generalised in connection with a primary amplitude (or hypersurface) of a Euclidean space $S_n$ of $n$ dimensions. In this hypersurface there are $n-1$ principal directions of linear curvature, and there are grades of curvature of successive spatial dimensions up to $n-1$. The orientations corresponding to the principal values of every special grade of measure of curvature are those determined by the principal directions of linear curvature.

Two chapters are devoted to surfaces existing in a region of the quadruple space. In this connection two kinds of linear curvature of the surface have to be taken into account. One of these is curvature relative to the region, and is due to the relative deviation of a geodesic of the surface and the regional geodesic tangent at the point considered. The other kind is curvature relative to $S_4$, already studied in the third section; and the difference between the two kinds is due to the curvature of the regional geodesic. A detailed discussion of the curvature properties of the superficial geodesics is given, including the determination of the principal directions and the principal values of the various curvatures. The general theory of regions is illustrated by its application to a particular case—the ovoid, which is represented by an equation of the second degree analogous to that of an ellipsoid. Lastly, a chapter is devoted to minimal surfaces and minimal regions. In the course of a fairly full discussion it is shown that, on any minimal surface, the four lines of curvature through a point consist of two perpendicular pairs, each pair bisecting the angles between the other pair. For a minimal surface contained in a region the sum of the principal values of the curvature of a superficial geodesic relative to the region is zero. And, for a minimal region, the sum of the principal linear curvatures at a point is zero.

The *fifth* and concluding section of the book is less geometrical in character, dealing with the theory of invariantive concomitants of all possible types belonging to curves, surfaces and regions in an $S_4$. Such concomitants possess parametric, orientational and local invariance under the limitation that magnification, distortion and disruption of the manifold are excluded. The method adopted for their construction is based upon Lie's theory of continuous groups of transformations. The author gives an aggregate of concomitants, algebraically independent of one another, in terms of which every intrinsic magnitude of the configuration can be expressed; and a geometrical interpretation of each concomitant of this aggregate is found.

Throughout the book the general theory is well illustrated by the solution of examples. The author's style is clear and attractive. If there is a lack of conciseness in many places it is due in part to the non-use of vectorial methods. The reader who is familiar with the simplicity of analysis attained in the more general geometry of Riemannian space by the use of vectors of $n$ components, tensors of various orders, and the summation convention for literal indices, may be impatient with the enormous number of equations and special symbols that occur, especially in the treatment of regions. But, having chosen his instrument of analysis, the author handles it skilfully and most effectively. Whatever opinions may be held about details of treatment, the fact remains that Prof. Forsyth has given us a most valuable book which, to quote his own words, "has grown in continuous development of its own from first to last, without occasion on my part to select or modify or incorporate the researches of other writers". It is a worthy successor to his other treatises, and should find a place on the library shelves of every university or seat of learning, and of every professional mathematician. C. E. WEATHERBURN.

**Bell's Practical Modern Arithmetic for Senior Classes.** By W. F. BEARD. Pts. I, II, III. Pp. vi+74, vi+74, v+74. Manilla 10d. each. Cloth, 1s. each. Answers, 6d. each part. 1931. (Bell)

These books are a good four-years' course for pupils in "Modern" and "Senior" schools: they presuppose a grounding in the fundamentals. Revision is well provided for—even Parts I and II begin with quite elementary operations. There is a good deal of mensuration—the cylinder appears at the end of Part III; this is also the stage of the course in which Simple Interest is dealt with. The author seems to favour, in subtraction, the "shop" method, and in multiplication, placing the left-hand figure of the multiplier under the right-hand figure of the multiplicand (any necessary decimal point being inserted after a rough check), and, in division of decimals, an integral divisor. The author recommends that those who use the books should have access to some larger work on Arithmetic. T. M. A. C.

**A Revision Arithmetic.** By E. H. LOCKWOOD. Pp. viii+176+viii. With or without answers, 2s. 6d. 1930. (Longmans, Green)

It is quite safe to say that any teacher of Arithmetic will be glad to see this book. The author says that its primary aim is preparation for the Oxford and Cambridge School Certificate Examination, and the book consists of a final revision course quite suitable for such a purpose. But Mr. Lockwood also says that he has had in mind the cause of "Scientific Humanism". Each chapter is therefore introduced by an exceedingly able and vivid account of its subject and it is suggested that these introductions should be really studied. The examples are all original, the material for them gathered from "newspapers and other every-day sources". In the preface the case is imagined of someone saying farewell to Mathematics on joining a Classical or Modern or History Sixth. One can carry that on by suggesting that, if such a person were to work through this book before saying farewell, he would be likely ever afterwards to retain a respect—even an affection—for the ancient science of Arithmetic. It is evident that Mr. Lockwood has two of the qualities essential for good teaching—a lively interest in his subject and a willingness to take trouble to make that interest effective. T. M. A. C.

**Four-Place Tables.** Unabridged Edition. By E. V. HUNTINGTON. Pp. 32+i. Cardboard covers. 2s. 1931. (Allen and Unwin)

These tables, originally published in the U.S.A., are now published for the first time in England. The angles are expressed in degrees and decimal parts of a degree instead of in degrees, minutes and seconds. A conversion table is provided for converting minutes and seconds into decimals of degrees and *vice versa*.

In addition to the usual tables of logarithms and trigonometrical functions there are tables of squares, square roots, cubes, cube roots, circumferences and areas of circles, logarithms of radians, exponentials and logarithms to base $e$.

In the sine and cosine table the numbers referring to the sine are printed in type different from that of the cosine. This practice occurs wherever a table can be read in two ways, thus limiting what is often a source of confusion. The pages are somewhat larger than is usual so that all the values of any one function from 0° to 90° can be found at one opening. Special tables are provided for small angles and angles near 90°. At the end there is a condensed table and a list of useful constants. By cutting along specially provided dotted lines the reader can make himself a thumb index. On the whole, much thought has gone to the making of these tables and they can be recommended. C. F.

**Geometry Test Papers.** By W. G. BORCHARDT. Pp. 88+viii. 1s. 1930. (Rivingtons)

A useful set of 50 graduated papers suitable for school use, including questions on Constructions and Numerical Calculations as well as Riders.

**Test Papers in Algebra and Geometry.** By JOHN DOUGALL. Pp. 40. 1s. 1930. (Blackie)

A set of 30 test papers for use by candidates preparing for Public School Entrance Scholarship Examinations.

**Thirty Tests in Elementary Mathematics.** By J. H. WELLS. Pp. 44. 8d. 1930. (Harrap)

These test papers are intended primarily for those preparing for the School Certificate and include questions on Arithmetic, Algebra, Geometry and Trigonometry.

**Middle School Test Papers in Mathematics.** By J. J. WALTON. Pp. 75. 1s. 3d. 1931. (Pitman)

25 papers in each of the subjects Arithmetic, Algebra, and Geometry, intended for school use in the year preceding that in which a scholar should take his School Certificate Examination.

**School Certificate Geometry.** By A. G. CRACKNELL and G. F. PERROTT. Pp. x + 344. 4s. 6d. 1931. (Univ. Tutorial Press)

This is a carefully written, well-arranged text-book for general school use, mainly on Euclidean lines, but embodying many modifications in accordance with modern teaching experience. There is no attempt to broaden the basis by appeals to intuition for fundamental facts. On the contrary, after stating and expounding the necessary definitions and axioms in the first chapter, the authors proceed to the deductive development based on proof by superposition. The book covers the subject-matter of Euclid, Books I-IV, and includes the more important propositions relating to Ratio and Proportion and Similar Figures based on the assumption that all the magnitudes involved are commensurable. There is no Solid geometry. The exercises are very numerous and include Drawing problems and calculations as well as a large collection of riders, and the last chapter contains some very useful help for the student in tackling riders. The work is to some extent based upon the *School Geometry* by W. P. Workman and A. G. Cracknell.

In some of the diagrams the lines are unnecessarily thick. A few of the definitions are not free from objection; for instance, "A rectangle is a quadrilateral in which each of the four angles is a right angle". This appears on the same page as "A square is a quadrilateral in which all the sides are equal and any one angle is a right angle". The following statement on page 264,

> "the fixed line $HK$ subtends a right angle at $A$.
> But the angle in a semi-circle is a right angle.
> Hence the locus of $A$ is a circle with diameter $HK$",

is scarcely up to the standard of logical accuracy which is generally a distinctive feature of this book. There is a misprint on page 310: for "$BD = BP$" read "$BD = DP$". In stating that two triangles are similar, the authors are not always careful to give the lettering in the right order.

W. J. D.

**Elements of Analytical Geometry.** Vol. I. Straight Line and Circle. By J. I. CRAIG. Pp. xiv + 415. 12s. 6d. 1930. (Macmillan)

This is a most unusual book: and its scope is far wider than is indicated by its title. Mr. Craig is lecturer in Pure Mathematics in the Royal School of Engineering, Cairo; and he has had to lecture to large classes, composed of students of very varying degrees of attainment. He has tried to satisfy "the intrepid explorer, the average wayfarer, and the patient plodder". Furthermore, he has had to provide for students of mathematics who take the science as a branch of a liberal education, and others who learn the subject for use in their professions. The book is accordingly divided into chapters of elementary theory, advanced theory, and practical applications. The theoretical student will find what he wants in the first and second; the practical, in the first and third.

It is evident that Mr. Craig has had to do everything for his students, the introductory chapter establishing geometrical, algebraical and trigonometrical data. Thus there are sections which summarise elementary geometrical propositions; algebraical results such as the progressions, the binomial expansion and the remainder theorem; eight pages are devoted to determinants and eight to elementary trigonometry. The treatment of coordinates is very thorough and includes consideration of various kinds of curvilinear coordinates. The chapters on the straight line include a full account of cross-ratios and homographic ranges; also of tangential and homogeneous coordinates, while there is quite a long section on vector algebra. The theory of the circle is treated with equal exhaustiveness.

The practical applications are varied indeed: all kinds of graphs and graph papers are referred to, as well as slide rules, nomography, linkages, railway curves and sound ranging.

There are numerous test papers at various stages, and a collection of "Studies" at the end, which are on the lines of questions in a general mathematics paper for Scholarships at Oxford or Cambridge—but somewhat longer.

To conclude this description, answers to the examples are *not* given, but there is an index, the book is well got up, and has been translated into Arabic.

But the question which exercises the reviewer is—who is going to read it outside Egypt? It is an interesting book for the mathematical library, and could be put into the hands of an advanced student on account of the unusual and interesting grouping of the subject-matter. But in England it is not necessary to combine into one book the needs of totally different classes of students: the mathematical specialists have their courses, and the technical specialists have theirs; while both courses are entirely different from that under discussion. Intrepid explorers, average wayfarers and patient plodders are separately catered for here. There is undoubtedly corn in Egypt, but it is not for export. N. M. G.

**Lehrbuch der Differentialgeometrie.** Bd. I. Kurven und Flächen im Euklidischen Raum. By A. DUSCHEK. Pp. viii+250. Bd. II. Riemannsche Geometrie. By W. MAYER. Pp. viii+246. Rm. 17 each. 1930. (Teubner)

When a region of mathematics which is important in itself is also part of a wider region, the problem of exposition is always difficult. Nothing is more unsatisfactory than to discuss the parabola without mention of the general conic, or than to pay attention exclusively to the axial and focal properties of the conic. On the other hand, the student to whom the parabola is only a special case seldom has its special properties at his finger-tips, and in a projective treatment the relations of a conic to its auxiliary circle are almost irrelevant and do not become familiar, while the eccentricity can hardly be mentioned. A similar difficulty has come over differential geometry since the tensor calculus became a subject with which every mathematician must have some acquaintance. The elements of the classical theory of a surface are vivid enough without reference to absolute derivatives and tensors, and an effort to teach them along the lines of ultimate development is apt to seem artificial. That this is a very real difficulty is seen by reference to the last two English works: there is not a word in Weatherburn's that prepares the reader for the tensor calculus, but no one would recommend Campbell's literally as an introduction to the classical theory.

More could not be said in recommendation of the latest German text-book than that its two authors have solved the problem triumphantly, in spite of the fact that in the collaboration each of them is responsible for one volume, and that we are told explicitly that the second volume can be read without any knowledge of the first. Dr. Duschek's volume is as easy to appreciate, as concrete, if the word is permissible, as Darboux or Scheffers, and yet the reader begins Dr. Mayer's volume with the preliminary difficulties of the abstract subject already overcome.

The only detail in which the first volume seems to me inadequate, within its range, concerns geodesic torsion. That this is a function of direction intimately connected with normal curvature was appreciated by Darboux, and in view of the general nature of his treatment it is really surprising that Dr. Duschek should not have laid stress on the tensor character of the coefficients of the second quadratic form and have associated this tensor as readily with a pair of distinct directions as with a single direction; he would then have recognised that with a pair of perpendicular directions the resulting scalar measures the spin of the tangent plane to the surface round the tangent to a curve, a spin which differs from the torsion of the curve precisely by the rate of change of the angle between the tangent plane and the osculating plane. The importance of the function would have been implicit in its tensor origin, and the interpretations of it would have been on that account the more interesting. Supplemented in this respect, Dr. Duschek's volume is definitely the best I know with which a student could begin.

The main lines of Riemannian geometry are still being laid down. The fundamental ideas of the tensor calculus form a necessary beginning, and there are certain extensions from classical differential geometry which are inevitable. But as Whitehead has remarked, every method of research creates its own applications; it is not until the immigrant applications are outnumbered by a native population and are themselves naturalised that a method can be said to have resulted in an independent region of mathematics. Riemannian geometry, in Dr. Mayer's account, has become such a region.

The first four chapters of the second volume explain the tensor calculus, introduce the metric which characterises a Riemannian space, and give the Frenet formulae. There are, I think, actual mistakes in the account of angles in a real twofold, since the value of the bilinear expression $g_{ik}\lambda^i\mu^k$ may be negative, and since also an angle if restricted to a definite range is not a continuous function. Moreover, the discussion could not easily be extended to the complex domain, since it depends on the choice of the positive square root of $g_{ik}\lambda^i\lambda^k$ as *the* length of $\lambda$, a choice which has no analogue, and which, as is shown at length in my *Prolegomena to Analytical Geometry*, has no advantages even in the real domain. This is, however, a detail, without reaction on the remainder of the work.

The fifth chapter deals with the calculus of variations and is beyond praise.

The sixth chapter introduces parallelism and geodesic manifolds, defines planes within a Riemannian space, and shows that spaces of constant curvature are essentially important, not a curiosity with which we happen to be able to deal readily. In the seventh chapter come submanifolds in general, with their curvatures relative to the containing manifold. A chapter on special spaces includes an account of the Cayley-Klein representation of a space of constant negative curvature, and the concluding ninth chapter considers the general submanifold of a space of constant curvature and studies the equations that correspond to those of Gauss and Codazzi in the simpler theory.

Almost the only evidence that Riemannian geometry has grown from an older root is in the persistent representation of curvatures in the unnatural form $1/\rho$, but since the references to the Erlanger programme are in the first volume the reader of the second cannot ignore the first without loss. The authors say that the two volumes are independent of each other. Certainly no student of differential geometry can afford to be independent of either of them.

E. H. N.

**Seven Place Natural Trigonometrical Functions.** By H. C. Ives. Pp. vi+222. 12s. 6d. 1929. (Wiley; Chapman and Hall)

As transcription of the whole of a lengthy title would reveal, this is not a book for school or college use. It is a pocket volume for surveyors, designed, in the words of the preface, "to meet a long-felt need for a compact field or office book". When we quote that "attention is especially called to Table XV

as likely to be of great service ", and explain that this table is not numerical but is a collection of elementary trigonometrical formulae, it will be clear that criticism from a *Gazette* point of view would be absurd. The tables of general interest are available in Chambers or Barlow. There are several small tables, for example one giving refraction corrections, and three connected with the position of the pole star, which are needed in field work, and there are five Appendices, occupying fifty pages, devoted to the elements of practical surveying. The book is cleanly printed in small type on thin but remarkably opaque paper, and has every appearance of being well suited to its special purpose. E. H. N.

**Non-Interpolating Logarithms, Cologarithms and Antilogarithms.** By F. W. JOHNSON. Pp. x+117. $2.25. 1931. (Simplified Series Publishing Co., San Francisco)

Five-place logarithms with five thousand entries on a double page ! How is it done ?

The basis is of course a grid of 50 lines and 100 columns, so that for example the rows from 600 tc 649 have columns from 00 to 49 on the left-hand page and from 50 to 99 on the right-hand page. The first pair of digits of a logarithm changes slowly, and a single entry, printed in very large type immediately to the right of the argument column, serves for a large block of arguments, whose precise extent is shown by " rules ", that is, black lines : the block for ·80 is delimited by two rules of which the upper crosses the 630 line between columns 95 and 96 and the lower crosses the 645 line between columns 64 and 65 ; in other words, every number from 6·3096 to 6·4564 has 80 for the first two digits of its logarithm, and this is indicated perfectly by a frame separating this group of numbers from those before and after. The second pair of digits of the logarithm is inserted once only, in the space between two columns, as soon as possible after the point at which it becomes effective—" as soon as possible ", for room for such insertion is provided not between every pair of columns, but only after columns headed 3, 6, 9 : in the 638 line, we find 52 in the space between columns 56 and 57, and 53 in the space between columns 69 and 70, implying that every number from 6·3857 to 6·3867 has 52 for the third and fourth digits of its logarithm ; thus before coming to the entries in the columns themselves, we know that these logarithms all begin ·8052. Finally, the entry in the column is simply the fifth digit of the logarithm, overscored if it is to be taken with an entry in the next available space instead of with an entry already in use : actually in the 638 line, column 56 has an overscored 0 and column 69 has a plain 9, showing that the range of numbers whose logarithms begin ·8052 is from 6·3856 to 6·3869 (a moment's consideration of an example will show that the overscoring is hardly necessary).

Briefly, as far as the first four digits are concerned, the table is what is called a critical table, that is, one in which the value of a slowly-changing function is recorded between the argument values between which it changes, but the typographical principle involved in the utilisation of spaces which are no wider than are commonly left to guide the eye is, that a crowding of figures which would be intolerable if the whole page was covered is inoffensive when only one space in three or four is occupied.

For definiteness the first table in the book has been described This fills 36 pages. There follow five-place cologarithms of a three-figure argument (2 pages), five-place antilogarithms of a five-figure argument (40 pages), and four-place logarithms, cologarithms, and antilogarithms of four-figure arguments (4 pages each). It is perhaps worth while to repeat that the tables are non-interpolating : four-place logarithms in four pages or two are common enough, but they are tables with three-figure arguments in which a fourth figure of the argument involves interpolation, and they are not to be compared with Mr. Johnson's. A thumb index contributes to the convenience of the book. E. H. N.

**Standard Four-Figure Mathematical Tables.** By L. M. MILNE-THOMSON and L. J. COMRIE. Pp. xvi + 246, with loose card. 10s. 6d. 1931. (Macmillan)

The tables are as follows: logarithms and antilogarithms, addition and subtraction logarithms (*i.e.* $\log(1+n)$ and $\log\{1/(1-n)\}$ with $\log 1/x$ as argument); square roots of $n$ and $10n$, cube roots of $n$, $10n$, and $100n$, powers as far as the fifth, reciprocals; $\log_{10} n!$, $0{\cdot}6745/\sqrt{n}$, $0{\cdot}6745/\sqrt{\{n(n-1)\}}$, $2\pi/n$; natural and logarithmic trigonometrical functions with angles in (i) degrees and tenths, (ii) degrees and minutes, (iii) thousandths of a radian, to $\frac{1}{2}\pi$, (iv) tenths of a radian, to 7·9 radians; natural and logarithmic hyperbolic functions and powers of $e$, for the ranges 0·000–1·000, 1·00–6·00, 6·0–15·0; multiples of $\log_{10} e$ and of $\log_e 10$; inverse trigonometrical functions in radians, inverse hyperbolic functions, and natural logarithms, for the ranges 0·000–1·500, 1·50–5·00, 5·0–15·0, 15–60; the gudermannian and its inverse, the gamma function and its logarithm, and the error function; coefficients for second-difference interpolation; circular measure in degrees and degree measure in radians; binomial coefficients, the first 560 prime numbers, sundry constants with their logarithms; proportional parts (a detached duplicate of this table printed on thin card is supplied with the volume).

The tables are for the most part to four decimal places, with a typographical device to render them serviceable to one-third of a unit in the last place. Apart from this device, they are of course almost all individually familiar, but many of them have been recalculated for confirmation. The tables of trigonometrical functions and inverse functions in thousandths of a radian are new, as far as I know, and are extremely valuable: the publishers claim that the table of natural logarithms also is new, but apart from the dot for the last one-third of a unit, an equivalent table was given by Smith and Longley in 1929.

Eight pages of explanation precede the tables, and at the end of the volume a collection of formulae, including a table of integrals, occupies twenty-six pages. A warning on $\int dx/x$ might have been useful, and the only people who use undetermined coefficients for computing partial fractions are those who do not know better, but it is not on such incidental text that the value of the book depends. The advice on the use of the actual tables could not have been sounder or more lucid.

There are two editions of the book, one in which the logarithms all have their true values, with negative characteristics when necessary, the other in which negative logarithms are increased by 10.

Although the inclusion of tables of $0{\cdot}6745/\sqrt{n}$ and $0{\cdot}6745/\sqrt{\{n(n-1)\}}$, which are factors in probable-error formulae, reminds us that a function may suddenly acquire an unsuspected widespread importance, it is difficult to imagine that the work of Dr. Comrie and Mr. Milne-Thomson can ever be superseded. The compilers' names are a guarantee of accuracy up to the very limit of human fallibility, and the printers of the *Gazette* have never produced better pages than those of this handsome volume. E. H. N.

**Georg Cantor.** By A. FRAENKEL. Pp. 78. Rm. 4.50. 1930. (Teubner)

The separate publication of this study of Cantor's life and writings, which appeared first in the *Jahresbericht der deutschen Mathematiker-Vereinigung* for 1930 (vol. 39, pp. 189-266), is very welcome. Dr. Fraenkel writes of the work with the authority of a master, of the man with the sympathy of a friend. He describes the development of the theory of aggregates from Cantor's earlier researches in the theory of numbers and in Fourier series, and he describes also the breakdown of Cantor's health, and the diversion of his immense intellect not merely to philosophical and religious speculations but to such an unworthy aim as establishing the Baconian authorship of Shakespeare's plays. He makes an attempt to trace the acceptance of Cantor's ideas into the mathematical thought of his time, and he deals at length with the relation of his later writings to those of others.

Considering that Cantor was claiming for mathematics a region that was not so much strange as hostile, the suspected source of the fallacies which Weierstrass was engaged in eradicating, the recognition of his genius was not remarkably tardy, but Cantor was too soon discouraged, embittered too sadly by Kronecker's antagonism to be maintained by Mittag-Leffler's loyalty and admiration. He lived to the age of seventy-three to know himself honoured for the work he had accomplished before he was forty.

Dr. Fraenkel concludes with a careful bibliography of Cantor's published writings and a list of ten memoirs dealing with him. E. H. N.

**The Foundations of Mathematics and other logical essays.** By F. P. RAMSEY. Edited by R. B. BRAITHWAITE. Pp. xviii + 292. 15s. 1931. (International Library of Psychology, Philosophy and Scientific Method. Kegan Paul)

F. P. Ramsey died in 1930, having acquired before the age of twenty-seven the reputation of being one of the most acute thinkers of his generation; we have now a collection of his logical and philosophical essays, published and unpublished.

From the bibliography we learn that the reprint of published work excludes only articles on economics, contributions to the *Encyclopaedia Britannica*, and a number of short reviews. One article in the form of a review is included: a long criticism of Wittgenstein's *Tractatus Logico-Philosophicus*. Not only is this on a different scale from an ordinary review, but Wittgenstein's was the greatest influence in Ramsey's mental development. Anyone who wishes to discover from Ramsey's papers his contribution to philosophical thought must first examine his response to Wittgenstein's teaching and eliminate what is derived directly from Wittgenstein; this is simple, for Ramsey is lavish with acknowledgments.

The editor tells us that of unpublished material he has excluded, apart from fragmentary or superseded notes and notes for a course of lectures, only a draft of the first four chapters of a general treatise on logic; this is quite unsuited for publication, and the author was himself profoundly disappointed with the work.

Of the reprinted papers, one was published originally in the *Gazette*. This is an account read to the British Association in 1926 of the development of mathematical logic subsequent to the publication of the first edition of *Principia Mathematica*; it is a model of what such an address should be. The difficulties associated with the axioms of infinity and reducibility are explained clearly, the theories into which Brouwer, Weyl, and Hilbert were driven by them are criticised, and applications of Wittgenstein's ideas by the author to the construction of an alternative theory are indicated. Ramsey's detailed account of his work in this field was published by the London Mathematical Society in the paper which gives this volume its name and its permanent value. The majority of writers on the subject, says Ramsey, have concentrated their attention on the explanation either of the concepts or of the propositions of mathematics, and supposed that a satisfactory explanation of the neglected category would immediately follow. That this supposition is erroneous he has little trouble in showing. His own contention is that mathematics consists of what Wittgenstein calls tautologies; there are difficulties in this theory, but every theory of mathematics bristles with difficulties, and Ramsey's theory at least removes some of those which caused the German critics of Russell and Whitehead to turn in despair to formalism on the one hand or intuition on the other.

Of the other reprinted papers, three are philosophical and one deals with a problem of formal logic. Most of us have known the stage of suspecting that formal logic is only practice in setting down in shorthand statements which must be tested by sheer inspection; to anyone still in this doubting frame of mind, this paper by a master of the technique may be recommended.

Of the papers hitherto unpublished, those on the nature of theories and on

causality are interesting, but the most provocative is one on truth and probability. Ramsey puts forward the theory that probability is derivable from partial belief, and that partial belief is measurable in terms of betting odds, this phrase being used to cover almost any quantitative comparison of alternative lines of action. As he recognises, allowance must be made for the extremes of prudence which would decline to bet at any odds if the stakes were high and of a sense of proportion which would refuse to be bothered with a bet of a trivial amount—in other words, the difficult problem of "risk", which takes its most notorious form in the St. Petersburg paradox, is assumed to be solved in advance; also the odds must be not those of the first comer or of the man in the street but of an "ideal person". Objections to the theory come to the mind at once. Who will bet on races that can never be run, for example, on questions of the origin of the solar system, although answers to such questions may surely differ in probability? Is it plausible that degrees of belief, measured in terms of actions to which they lead, form a compact aggregate? More fundamentally, does not the incursion of the "ideal person" imply that the relations to be discussed are between propositions alone in spite of the attempt to introduce a subjective foundation?

The author's first defence of his theory is that it gives a clear justification for the axioms of the calculus of probability. "If partial beliefs are consistent they will obey these axioms, but it is utterly obscure why Mr. Keynes' mysterious logical relations should obey them." To which one might reply that as a matter of experience human beliefs seldom are consistent and that it is "utterly obscure" why we know as logicians that those of the "ideal person" must be. But there is a much simpler reply, which Ramsey might have found in the series of papers by Wrinch and Jeffreys published long before his article was written, and made the basis of Jeffreys' recent *Scientific Inference*. All that is intrinsic to the probability relations is the existence of relations of greater than, equal to, and less than, together with the uniqueness of the maximum probability corresponding to certainty and of the minimum probability corresponding to impossibility. The axioms are secured not by obedience on the part of the probability relations, but by skill on the part of the mathematician when he assigns measures to probabilities, a task in which he has as much freedom as in assigning measures to beliefs. One can believe in the mathematician who adopts conventions which lead automatically to a simple calculus. The "ideal person" who, in advance of calculation, knows that it would be reasonable to choose one of two alternative lines of action on precisely 3001 occasions out of 8009 seems to me utterly fantastic. Whatever the difficulties of other theories of probability, the subjective treatment only adds difficulties of its own.

How artificial this subjective treatment becomes is clear in one of Ramsey's own examples. "What habits in a general sense would it be best for the human mind to have? . . . Let us take a habit of forming opinion in a certain way; *e.g.* the habit of proceeding from the opinion that a toadstool is yellow to the opinion that it is unwholesome. Then we can accept the fact that a person has a habit of this sort, and ask . . . what degree of confidence it would be best for him to have that [yellow toadstools] are unwholesome. And the answer is that it will in general be best for his degree of belief . . . to be equal to the proportion of yellow toadstools which are in fact unwholesome." Since elsewhere we are told to estimate beliefs by actions, this actually means, according to Ramsey's use of words, that if half the yellow toadstools in the world are poisonous, a man can form no better habit than that of eating one of every two which he sees. Would the human race have survived the formation of habits like this? If Ramsey's theory cannot be true unless every reasonable man would answer this question in the affirmative, the theory is self-condemned ($p \,.\, p \supset \sim p : \supset \sim p$) unless indeed the theory of types invalidates the argument. E. H. N.

**Scientific Inference.** By HAROLD JEFFREYS. Pp. 247. 10s. 6d. 1931. (Cambridge University Press)

This book is meant mainly for theoretical and experimental physicists. The author himself says so, though he lays no great stress on the statement. It is a most stimulating and interesting account of how to build up an ordered structure of "scientific knowledge" from the separate pieces we "learn by experience"; it can be recommended to anyone who (*a*) has a moderate knowledge of mathematics, (*b*) is ready to think a bit for his amusement.

The early chapters on probability and quantitative laws enunciate two main theses which occur again and again as the book develops; they are "the notion of probability is fundamental in scientific inference" and "the more simple the law the greater is its prior probability". The mathematical discussion of probability is *ab initio* and is, to me at any rate, one of the most difficult bits to grasp in the whole book. Dr. Jeffreys appears to attach a meaning to the statement "the probability of $q$ given $p$ is either greater than, equal to, or less than that of $q'$ given $p'$" *before* he expresses probabilities by means of numbers. This is one of the points where the reader may find considerable amusement if he cares to pay for it. Personally, I could not see eye to eye with the author, but I ended a very enjoyable mental ramble with two conclusions; (i) Dr. Jeffreys had a clear concept in his mind where the present reader has a fuzzy apprehension, (ii) the reader can get round the particular point in his own way and go on with confidence to the developments which follow.

A chapter on errors rounds off the discussion of probability, and this is followed by chapters on physical magnitudes, mensuration, Newtonian dynamics, light and relativity. These make excellent reading. The first two of them, on physical magnitudes and mensuration, show how the applied mathematician, when considering physical objects, can carry over results obtained by pure mathematics without any essential break in the continuity of his argument. It was a point that had always defeated me. There are two principal places where the discontinuity may occur. The first is irrational number, for the average pure mathematician one thing and for the average physicist something quite different; the second is analytical geometry, which again means different things to different people (not to mention those of us who know how it is done and are not much concerned with what it means). The discussion of physical magnitudes makes it clear what Dr. Jeffreys means by $\sqrt{2}$, though I wish he would not call irrational numbers "fractions". The chapter on mensuration is based on measurement and not on Euclid's postulates, and is brought up to a point which enables one to develop, from what has gone before, all the usual analytical geometry of rectangular Cartesian axes; there is no "just assuming that geometry applies".

The two chapters on Newtonian dynamics and on light and relativity are equally attractive. At the risk of turning this review into a series of confessions of ignorance, I acknowledge my indebtedness to the book for a glimpse of the meaning of the numbers which occur in the theory of relativity. As a piece of pure mathematics which juggled out the right answer, I knew more or less what the theory was; as a piece of applied mathematics, I seem at last to have got some definite meaning into the variables. Here and there the exposition assumes a previous knowledge of the details of some experiment; notably the Michelson-Morley experiment can scarcely be grasped from the introduction and description given. To quote one of the author's amusing chapter headings, "You mustn't leave out so many things".

The book ends with a chapter in which the author explains why, at certain points, his theory diverges from other theories of scientific knowledge.

You may not agree with Dr. Jeffreys but, if you read the book at all, you will be obliged to confess that he has written an excellent account of his own solution of a problem, one with considerable bearing on his own researches, "why can we apply physical laws far beyond their original range of verification both in time and in distance?"

W. L. F.

**Mathematische Statistik der Personenversicherung.** VON DR. GUSTAV ROSMANITH. Pp. 141. Rm. 8 geb. 1930. (Teubner)

As the rate of mortality is a function of many variables, period of observation, race, geographical situation, sex, age, occupation, and that for insured lives being different from that for the general population, with even the varying classes of insurance showing different mortalities, it will be seen that the construction of mortality tables is a never-ending process, and that it is therefore necessary to have a constant supply of young men able to understand the construction and graduation of the more important tables, and to be aware of their chief characteristic features.

It is therefore with great pleasure that we welcome such a production as this by Dr. Rosmanith, who, as he states in his preface, does not set out to give us anything essentially new, but brings much old material into a well-ordered form suitable for beginners. In this he has been eminently successful, for there is no other book known to the reviewer which brings the material into so small a compass and into such a clear form. The book should save the student much wandering amongst the sixty volumes of the *Institute of Actuaries' Journal*.

Besides explaining the essential features of the principal British tables, and those of Germany and Austria, the author gives information about the collection of data, and shows the method of construction and graduation of the tables. The work should be of use to all engaged professionally in mathematical education, who should have some idea of the various applications of their subject.

Being a first edition the book contains some misprints, and it is not quite correct to say that the mortality table of the twenty English Offices shows a mortality of 12% in the first year of life, because this table only starts at age 10. Dr. Rosmanith has obtained the 12% from the table given at the end of his book, which is a copy from King's text-book, and it should be pointed out that the earlier portion of this table was a graft by Mr. King from material collected elsewhere.

On page 132, William Shanks is called Stanks; on page 78, Lambert is correctly given and also as Lampert; p. 79, table column 2, 9025 should be 8025; and p. 93 (in the table), ·018497 should be 028497; and in the last column +133 and −545 should be −133 and +543. Other mistakes occur, but will not affect the reader.

The appendix might perhaps be better as a prefix, for without the knowledge there given the reader's understanding of the rest of the book would not be very satisfactory. W. S.

**The Theory of Functionals, and of Integral and Integro-differential Equations.** By VITO VOLTERRA. Translated by M. LONG. Pp. x+226. 25s. 1930. (Blackie)

There are certain stages in mathematical thought in which the point of view is suddenly enlarged. In analysis these stages have been (i) from the vagaries of particular numbers to the consideration of a continuous variable, (ii) from particular functions to an attack on the general function introducing calculus and the theories of functions of the complex and of the real variable.

Volterra's life-work has been the exploration of the next stage—that in which particular operations on general functions are replaced by general operations which are called Functionals.

$U[f(x)]$ is a functional of the function $f(x)$ if, whenever *all* the values of $f(x)$, $a \leq x \leq b$, are given the value of $U$ is determinate, as for example,

$$U = \int_0^1 f(x) f(1-x)\, dx.$$

Such simple operations as $f^2(x)$ are also functionals, but of so degenerate a type as scarcely to deserve the name. The typical functional depends on the

*simultaneous* values of $f(x)$ for all points, $x$, in an interval; it is, indeed, a function of which the "variable" is a point, not in ordinary, but in "function" space. Such a change in the point of view possesses several advantages. It enables one to solve problems such as those of integro-differential equations or of hysteresis by analogy with similar problems in ordinary function theory; it also ties together a number of apparently distinct branches of analysis. Lastly, it raises interesting new problems where the above-mentioned analogy breaks down. It is true that there are also some disadvantages. A worker in the Theory of Numbers must regard the uniform haze of algebra as a poor substitute for the sharp distinctions between individual numbers. Similarly, many an analyst will regard the theory of functionals as a desert in comparison with the theory of the properties, for example, of the Riemann Zeta function. But there is room both for the geographer and the astronomer.

In connection with the present book the reviewer is astonished that more interest has not been shown in England and Germany in the subject of functionals. Apart from the substantial and pioneering researches carried out by Hadamard and Volterra, each interested in the field because of its necessary connection with mathematical physics, the work has been carried further by comparatively few individuals, among whom we may mention M. Fréchet and G. C. Evans. This is all the more astonishing in that Volterra's book on Functions of Lines was published in the Borel series in 1913 when interest in Integral Equations was still at its height. It might have been expected that after the fundamental work by Volterra and Fredholm on linear functional equations there would have been an attack on the more general type of functional. Instead, there has been practically a slump. Even in mathematics the grand generalization has been given up for the vivid particular.

Volterra's *Theory of Functionals* is recommended particularly to three classes of reader, firstly, to the mathematical physicist—in the new quantum mechanics, for example, the Hamiltonian is replaced by a functional and the Hamilton derivative is simply Volterra's functional derivative. Also it may be mentioned that there is an intimate connection between Heaviside's calculus and Volterra's "composition". Secondly, the analyst will find that much of the theory of Dirichlet series is translateable into terms of Volterra's theory of the closed cycle. Lastly, those in search for new mathematical interests will find here room for considerable future expansion. There are also many parts of the general theory which should be placed by someone on a rigorous "epsilon" basis. It is probable that such an attempt would reveal unsuspected and interesting difficulties. Several new branches in so vast a subject may also occur to the reader.

The book opens one topic after another which cannot be exhausted in so brief a space. Indeed the reviewer feels that the book is not as much of a connected logical whole as it might be. It tends a little to the encyclopaedic style. But very full references are given and the reader will be stimulated to read many of these.

After an introduction with the definition of differentiation and integration—the latter topic only in a primitive state at present—the book proceeds to integral equations and then passes to the field corresponding to that of functions of several complex variables. Volterra then builds up the fascinating algebra of "composition" and, after some development of a more general analysis, closes with a few of the innumerable applications of the theory to physics.

The reviewer cannot let the occasion pass without thanking the publishers, Blackie and Sons, for their enterprise in making available to the English public so much of the work of Continental mathematicians. P. J. D.

**An Elementary Treatise on Actuarial Mathematics.** By HARRY FREEMAN. Pp. xiv + 399. 25s. 1931. (Cambridge University Press)

This book supplies a real need, that of providing, in one compact text, material for teaching and study for Part I of the Examinations of the Institute

of Actuaries. Hitherto candidates have had to spread their reading over prescribed parts and selected passages from different standard treatises, a mode of study which has its points, but is hardly ideal for the inexperienced beginner. The book under review gathers together, in less than four hundred pages, a surprisingly large amount of theory, advice and examples on Finite Differences, Summation, Differential and Integral Calculus, and Probability. It was also a happy thought to open with a chapter on Trigonometry.

The book seems to us well suited to its purpose, and the Council of the Institute has clearly been fortunate in inviting Mr. Freeman to write it. The chapters on the Calculus occupy about half of the book, and contrive to be fresh and helpful, while following the standard canons. The treatment of Probability is eclectic, a number of different standpoints and definitions being put forward, and the reader being presumably expected to develop his own philosophy—not a bad course, in a world which has seen many alternations of dogmatism and scepticism in this protean subject. The theory of mathematical expectation is prominent, but inverse probability finds no place.

In the Interpolation we feel that greater emphasis might have been laid on Newton's formula of divided differences and the simple but very effective "zigzag rules" which interpret it on a table; also that Sheppard's central difference notation might have been used more freely. In the Numerical Integration the Euler-Maclaurin formula is derived by an uncompromising use of inverse operators, as in Boole's "Finite Differences"; it would have been in order to enlarge on the real nature of this formula and the remainder term. Personally, too, we have always felt that the derivation of Weddle's Rule by a slight "faking" of the fifth Newton-Cotes formula, though historic and ingenious, is unnatural; the true derivation lies in a linear combination of a 7-ordinate Simpson with a 7-ordinate Three-Eighths formula in the ratio $9:-4$, which eliminates the principal error terms of each. But this point is not important.

The book is beautifully printed and pleasing to the eye, and to all appearance free from typographical faults. There seems to be some discrepancy, however, in the numerical illustration at the top of p. 35.

**Numerical Mathematical Analysis.** By J. B. SCARBOROUGH. Pp. xiv+416. 25s. 1930. (Baltimore, Johns Hopkins Press; London, Humphrey Milford)

This book covers very much the same ground as Whittaker and Robinson's *Calculus of Observations*; its scope is more restricted, but its methods are more detailed and in general more elementary. It is quite evidently the fruit of considerable practical experience on the part of the author in coping with all kinds of numerical situations, and the opening chapter, on numerical accuracy, makes explicit statement of conclusions arrived at by most practical calculators but hardly ever seen in print. The remaining topics are interpolation, numerical integration, solution of algebraic and transcendental equations, numerical solution of differential equations, the normal law of error and the principle of least squares, precision of measurements, empirical formulæ and harmonic analysis.

The best part seemed to us to be the treatment of the solution of differential equations, a careful and interesting discussion of methods associated with the names of Adams, Picard, Runge, Kutta, F. R. Moulton and W. E. Milne. The treatment of the normal law of error and the method of least squares adheres too closely to the classical Gaussian presentation and notation, in which the modulus of precision rather than the standard deviation is the parameter. Other statistical distributions find no mention.

The chapters on interpolation are distinctly disappointing. The author in his preface, "in order to make the book . . . as readable as possible", renounces any intention of using symbolic operators or divided differences. This self-denial defeats its own purpose most notably. For example, Stirling's central difference formula is derived from the Gregory-Newton advancing

formula by a sequence of numbered steps exhausting most of the alphabet and four large pages, and Bessel's formula fares almost as badly. Now Newton's divided difference formula, which Newton himself regarded as the *rem omnem* of polynomial interpolation, requires no more than the Remainder Theorem of elementary algebra for its proof, and leads to a tabular rule which is not so much a mnemonic as a luminous synthesis. From it we can write down at once any number of formulae; in particular we can write the Gauss forward and backward formulae, and by averaging or subtracting them obtain the Stirling, Bessel and Everett formulae in a line or two; in any case, either of the Stirling or the Bessel formulae can be obtained by differencing the other.

In the chapter on numerical integration much care and space is devoted to the separate determination of the remainder terms for the various formulae. Quite a proportion of this is unnecessary, for, as Dr. W. F. Sheppard pointed out over thirty years ago, the Simpson, Three-Eighths, Weddle and a number of other rules are linear combinations of trapezoidal formulae involving ordinates equidistant over the range, but at different intervals, and once we possess knowledge concerning the simplest formula we deduce at once the remainder terms for the others. One feels, too, that the author is a little unsentimental in urging the early interment of the Three-Eighths rule; even if it is inferior to Simpson's parabolic rule it will always arise in any discussion of the Newton-Cotes formulae, and it is rather interesting that the formula derived from a polynomial of degree $2n+1$ is less accurate than that derived from the one of degree $2n$. As a counter-motion, the present reviewer suggests a close season for that alternative to least squares, Mayer's method (p. 358) of pooling observational equations.

The solution of algebraic and transcendental equations confines itself to the methods of false position, graphical approximation, iteration by Newton-Raphson in one and several variables and, in a separate chapter, Graeffe's root-squaring method. The treatment is painstaking and thorough, and always clear. Plurality of complex roots causes trouble, of course, and Runge, Bairstow and Carvallo are cited for extraneous study. The neat and effective extension of Græffe due to Smeal and Brodetsky (*Proc. Camb. Phil. Soc.*, 1924), by which any number of pairs of complex roots are found unambiguously and the modulus and sign of real roots checked incidentally, has evidently escaped notice.

The chapter on harmonic analysis gives the 12- and 24-ordinate schemes due to the late Carl Runge, which are now fairly familiar, in the original or derivative forms.

It will be seen that this is a book which possesses many good qualities and some conspicuous defects. High praise must be given to the careful explanations, the numerous and well-thought-out examples and the sound precepts which are interspersed with the theory; but regret must be felt over many demonstrations which might have been much shorter, and yet more truly elementary.

The printing and format are very good, and the number of misprints that have escaped notice in revision is comparatively few. We may note a few: p. 45, first word; p. 104, at foot; p. 154, last term of (47 : 3); p. 156, second line; p. 228, seventh line; p. 290 (89 : 5); p. 366, Ex. 2, fifth value of $y$; and a few misplaced or absent accents in the citations. A. C. AITKEN.

**Text-Book on Compound Interest and Annuities-certain.** By the late RALPH TODHUNTER. Third edition, revised and enlarged by R. C. SIMMONDS and T. P. THOMPSON. Pp. xv+262. 17s. 6d. 1931. (Printed and published for the Institute of Actuaries by the Cambridge University Press)

The author of this work—a nephew of the once well-known Isaac Todhunter—had the fourfold qualification that he was a highly-trained mathematician, a very able practical and theoretical actuary, an experienced tutor, and the possessor of the gift of writing clear and elegant English. It was

therefore natural that the first edition, published thirty years ago, should take its place as an admirable and authoritative treatise, and that subsequent editions should not call for any very considerable organic changes. The revisers of the third edition, themselves experienced actuaries and actuarial tutors, in their preface very gracefully "express the hope that, despite the changes that have been made, the book will continue to be recognized as the work of its real author and to be known by his name". Unlike many—perhaps most—works on the subject, the book does not consist merely of formulae with cut-and-dried demonstrations, but it is really a *treatise* in which the subject is fully discussed and explained in a thoroughly practical way. The original edition contained a fair number of practical examples and illustrations, and the revisers have enriched the present edition by including a much larger number of fully-worked examples, many of them "taken from real life". They have also added a new and valuable chapter on the practical aspects of Loans repayable by means of Accumulative Sinking-funds, and have in general amplified the treatment, "where necessary, by the addition of comments designed to elucidate difficult points, and by the introduction of references to new work published since the Second Edition of the work was prepared". They have also made certain improvements in the order of the chapters, and sundry small changes of language. The chapter in the first and second editions on "Formulas of the Infinitesimal Calculus"—intended for young actuarial students with insufficient preliminary mathematical training—has been omitted, since the subject of that chapter is now more fully dealt with in the recent companion-volume, *Actuarial Mathematics*, by H. Freeman, F.I.A. (see Review, *ante*, p. 60). The useful compound-interest tables at the end of the book have been re-arranged and extended. They now include a rather wide range, viz.

$$(1+i)^n,\ v^n=(1+i)^{-n},\ s_{\overline{n}|}=\frac{(1+i)^n-1}{i},\ a_{\overline{n}|}=\frac{1-v^n}{i},\ \text{and}\ a_{\overline{n}|}^{-1}=i+s_{\overline{n}|}^{-1},$$

for $n=1$ to 50 and $n=60, 70, 80, 90, 100$, and for $100i=1,\ 1\frac{1}{4},\ 1\frac{1}{2},\ 1\frac{3}{4},\ 2,\ 2\frac{1}{4},\ 2\frac{1}{2},\ 3,\ 3\frac{1}{2},\ 4,\ 4\frac{1}{2},\ 5$ and 6. There are three pages of useful supplementary tables.

The revisers' work has been well done, as their names would guarantee, and the work is to be entirely commended as an indispensable text-book for all who wish to study the subject seriously. The new edition has been entirely re-set in type rather larger and clearer than that of the earlier editions, and the printing and general get-up are a good example of the high standard of the Cambridge University Press. G. J. LIDSTONE.

**Von Zahlen und Figuren : Proven mathematischen Denkens für Liebhaber der Mathematik.** VON PROF. HANS RADEMACHER und PROF. OTTO TOEPLITZ. Pp. vi + 164. 9.60 gold marks. 1930. (Berlin: Julius Springer)

This is an interesting, original and well-written book, likely to be useful to those who have a "feeling" for mathematics, and, nevertheless, but little technical knowledge of its processes. By careful selection of material, the authors have put together a series of chapters explaining various fundamental mathematical ideas, which are such that the non-expert, equally with the student of mathematics, can derive some measure of enjoyment. It is quite impressive to see how successfully the authors are able to treat such ideas as prime numbers, transfinite numbers, irrational numbers . . . (to take but a few examples), relying only on the most elementary logical ideas and general common sense, such as even the most unmathematical person must possess. It is interesting to have a book of this kind on one's shelves, for it demonstrates clearly that the ideas of mathematics are the ideas of logic and that the structural characteristics of mathematics are but a highly developed form of the common sense upon which even the non-mathematician, if he considers himself an intelligence at all, must rely in everyday life and in his more technical studies. D. M. W.

## CORRESPONDENCE.

### TWO POINTS IN THE TEACHING OF ELEMENTARY GEOMETRY.

[The opinion of teachers on the points raised by Prof. Lodge's letter would be of great interest and a reply form is enclosed in each copy of the *Gazette*. These should be sent to Professor Lodge, 330 Banbury Road, Oxford; they will then be classified and a report published in the *Gazette*.]

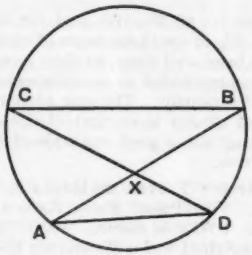
To the Editor of the *Mathematical Gazette*.

SIR,—(1) I wish there could be a consensus of opinion as to which area propositions should be considered as fundamental in the First Book.

Of course, the areas of congruent triangles are equal, and incidentally, therefore, the diagonal of a parallelogram bisects its area.

After that, to my mind, the fundamental proposition should be that the areas of parallelograms on the same base and between the same parallels (of which the base is one) are equal in area. All the rest are immediate deductions. The order does not much matter, but for its importance in practice the second theorem should be that triangles on the same base and between the same parallels are equal in area, proved by completing the parallelograms of which the triangles are each obviously half. There is a tendency to make the rectangle the basis and to equate parallelograms to rectangles, and triangles to half-rectangles, so as to bring in the mensuration "base times height" business. My own feeling is that we should start with the more general propositions above as the foundation, with mensuration as a corollary or consequence. The *Geometry Report*, pp. 50, 51, strikes a double note; but a consensus of opinion seems desirable.

(2) Do teachers insist on their pupils quoting congruent or similar triangles with the angular points given in *corresponding order*, and preferably one below the other? Thus in the figure:




The triangles $\begin{vmatrix} CBX \\ ADX \end{vmatrix}$ are similar in that order; hence $\frac{CB}{AD}=\frac{BX}{DX}=\frac{CX}{AX}$ if all three fractions are wanted, the numerators being taken from the first triangle in any order, and the denominators, *in corresponding order*, from the second triangle. If only two fractions are needed, we can either take numerators from one, and denominators from the other, as above, or one fraction from one triangle, and the corresponding fraction from the other, thus:

$$\frac{CX}{BX}=\frac{AX}{DX}.$$

In any case, the quoting in order of equal angles makes the taking of equal ratios definite and orderly without even having again to look at the figure.

I would be pleased to receive replies, assort them, and report on the result.—

ALFRED LODGE.

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